Revista Mexicana de Análisis de la Conducta,
1976, Vol. 2, Núm. 2, pp. 176-189.

Observations on Early Mathematical Behavior Among Children: «Counting».[1,2]

W. N. Schoenfeld, B. K. Cole, and D. M. Sussman[3]
Queens College (City University of New York)

ABSTRACT

This paper describes the first findings of a research project concerned with the analysis of mathematical behavior, the necessary conditions for such behavior to emerge in its rudimentary aspects, and the study of more complex mathematical skills.

RESUMEN

Este artículo describe los primeros hallazgos de un proyecto de investigación sobre el análisis de la conducta matemática, las condiciones necesárias para que dicha conducta surja en sus aspectos rudimentarios y el estudio de las habilidades matemáticas más complejas.

[1] The work reported here was done in part with the assistance of Grant MH-12964 from the National Institute of Mental Health. For their cooperativeness in making the project possible, we owe thanks to the New York City Board of Education, to Dr. M. Ehrlich (then Principal of P.S. 101), to Mrs. A. Feldman (Director of the Early Childhood Education Program), to Mrs. M. Murphy (Supervisor of the ECEP district containing P.S. 101), and to the teachers and paraprofessionals of the three pre-K classes from which our subjects were drawn.

[2] The substance of this paper was given as the Presidential Address of Division 25 of the American Psychological Association during its annual meeting at Chicago, Illinois, in September, 1975.

[3] The field work described here was done by WNS and BKC. Earlier on, before his untimely death, Dr. John Farmer was involved in the initial interest and thinking that led to the work. Later, DMS joined in the design of the research now projected as follow-up with both child and animal subjects, and also joined in the preparation of the present report.

The work reported here constitutes the initial effort in a projected research program having three goals: to begin an analysis of mathematical behavior, to reveal some of the conditions necessary for such behavior to emerge in its rudimentary aspects, and to plot a course into the study of more complex mathematical skills. It was hoped that a contribution could be made to an accurate description of the component behaviors that comprise mathematical skills, and to the development of teaching techniques that would promote efficient acquisition and lasting retention of those skills.

Aside from our personal interest in mathematics, the subject seems to call for a proper behavioral analysis today for several reasons. Among these are, first, the fact that mathematics is of great practical significance everywhere in the modern world; efficient training methods to establish and maintain mathematical skills in a learning population are therefore of equally great significance. Second, the repertoire of behaviors required for elementary mathematics, while it might be far from simple, nevertheless promises to contain fewer differentiated response components than the repertoire required for other forms of language. If so, this limited range of responses would simplify the problem of establishing congruent behaviors in laboratory animals comparable with human performances and therefore be of possible aid in the analysis of human performances. Third, the mathematical repertoire is quite similar the world over; that fact alone increases the inter-cultural generality of findings secured in a single culture.

By the term "mathematical behavior" we mean to include more than the customary approach of the schoolroom to the teaching of mathematics. That approach specifies the behavioral outcome of a response or response sequence as a "right answer" or "wrong answer". But it neglects to analyze the behaviors which precede and result in a "right answer", and upon which the "correctness" of that answer depends. Objectively speaking, the response sequences that compose any behavior pattern are never "right" or "wrong", but are rather the results of controlling variables which lead inexorably to one behavior or another. By attention to the actual behavior sequences, rather than inferring their nature from the "answers" we obtained, we sought to analyze their components and ultimately to describe the variables which produce a "right answer" instead of an "error".

Several considerations dictated how we approached our work in the field. First, since we aimed partly at developing procedures for training mathematical skills in individual learners, we focused on single subjects. Second, it seemed best to begin at the most elementary level of mathematical performance we could positively identify, and for which we could find subjects without overmuch difficulty. Third, we believed that reinforcement schedules would prove an important factor in our work because the maintenance of mathematical behavior once learned would be crucial for advance to later more complex skills; failure to advance could be attributed to only two sources, namely, failure to establish the proper base behavior needed to advance the repertoire, or failure to meet the conditions required to maintain the base behavior when the advance is undertaken; the latter point suggested to us that reinforcement schedule specifications would have to be a key feature of any total teaching program.

The decision which emerged from our thinking up to this juncture was to begin our work with a part of number behavior which is both elementary and comparatively limited in complexity, namely, "counting". We decided to work with the number range 0-9. The performance of "counting" requires more analysis, however, than would seem to be the case at first glance. Surveys of the literature, and of teachers' training and classroom teaching strategies, reveal ambiguities and confusions regarding even what is included in the behavior called "counting". In reality, there are many different behaviors and response sequences all of which are designated by this same term by laymen, by educators, and by psychologists. The different performances that are all called "counting" range from the rote recitation of number names, through simple number recognition, through object enumeration, to addition and subtraction. We aimed to observe the early stages in such a developmental sequence since later general "mathematical behavior", we presumed, must develop out of those stages. We chose as our subjects young children, pre-kindergarten or younger if possible, with the thought that it might be less difficult to disclose with them the rudimentary behaviors that lie within the observably elaborate, but hitherto undocumented, mathematical repertoires of children at more advanced stages of schooling.

The setting of our field work was a public grade school (p.s. 10) in Manhattan, New York City. We worked in whatever classroom chanced to be unoccupied on any given day, and the room was changed frequently on an unpredictable schedule. Our school is one of the older school buildings in the New York City system, serving a student body of about 1200 children with a teaching staff of about 53, including regular teachers and permanent substitutes. There were three pre-kindergarten classes functioning at the school under a special State grant, and our subjects were drawn from these classes. The school ranged from these pre-K groups through the 5th grade, all being in attendance from 9 A.M. to 3 P.M. The school population ethnically was approximately 44% Black and 56% Puerto Rican, and the school's location on the border between "Spanish" Harlem and "Black" Harlem made it reflect the features of an ethnically divided school in an economically and socially "disadvantaged" area. Within two months before our arrival, three individual teachers on separate occasions had been stopped and robbed on the street just outside the school doors, and the wariness of the staff about carrying money, and about personal safety, was quite marked. A school "security system" had been instituted, and was staffed by community volunteer mothers who screened visitors at the school's door (only one door into the building was kept operational) and helped prevent disruptive or malicious intrusions. Occasionally an intruder did infiltrate the school, and word of that circulated swiftly, to be followed by a general hunt by the teaching and custodial staffs. Community concern and involvement with the school was highly organized, and co-operative parent groups helped to maintain the educational atmosphere of a reasonably good school environment.

In the classes from which our subjects were drawn, a limit of 20 pupils per class was set. Admissions and attendance were voluntary in these classes. At the time of our contact, the Early Childhood Education Program of which our three prekindergarten classes were part had been operational for some 6 years. It seemed to have been used mainly by parents who wished their children to gain whatever "headstart" advantages ECEP might offer, or who needed to be free for employment, or both. Our reputation was good enough from the earliest days of our work, so that parents did not challenge us as "experimenting" with their children. In fact, several parents we talked with had heard about our work on some grapevine we did not know, and expressed gratitude for the extra effort being made with their children. We informally welcomed all parents to observe us at work with their children, but it turned out that most of their inquiries were in the form of seemingly casual, and somewhat shy, school corridor conversations. The informality of our invitations to parents was advised by the school principal who thought we might be overwhelmed by requests to work with all sixty children in the three classes instead of the sample of seven we used.

Each class in the pre-K program was handled by one teacher and two paraprofessionals. The three class teachers were white; of the paraprofessionals in each class, at least one was fluent in Spanish as well as English, The age range of the pupils in these classes was 4-5, that is, the year priori to the usual kindergarten age at the school and in the New York City system generally. The City Board of Education provided some syllabus and curriculum recommendations for pre-K classes, but teachers were left to their own devices in planning the class work and activities, since the Board did not evaluate the progress of pre-K children. Thus, the three pre-K teachers with whom we dealt varied more than might have otherwise been the case in conduct of their classes and in the educational materials they selected both in the form of games and of standard instructional implements. There was some convergence among the three classes, however, for several reasons: the similarity of materials as supplied by the board of Education; the need to function as part of a standardized grade school; the specialized pre-K training of the class teachers and paraprofessionals; and, the curricular guidelines of the early grades, kindergarten and post-K, with which the teachers and paraprofessionals shared a familiarity. The class routines and schedule were also structured to some degree by the three eating periods which formed part of the pre-K schedule. Food was brought to the rooms for a mid-morning cold snack, for a hot lunch, and again for a mid-afternoon cold snack. The noon meal was followed by an obligatory rest-nap period. Throughout our time at the school, the teachers and paraprofessionals were cooperative with us. They were curious about our work and asked whether they might monitor our procedures, but they never actually got around to doing so despite our open and repeated invitations. They did solicit information from us about the subjects we were handling, but it seemed they did so to support their own opinions about the potential and development of the children they had drawn from their individual classes for us to work with. Each teacher had earlier selected the children from her class who were to be subjects; they had made their choices for different reasons: one tried to give us "bright" children, another to give us "difficult" children, and a third to give us "typical" children. Although no teacher ever came to observe us, they seemed to value the information we gave them about each child based upon our own interactions with the children.

Our working periods with the children were brief and somewhat standardized in sequence and content: (1) preliminaries to the actual instructional periods, usually involving exploration of any changed room for us to work in that day; (2) the teaching period itself in which one or more "number tasks" or "number games" were reviewed and/or taught; and, (3) a play period or game period which was the "reinforcement" to the child for working with us on the "number task" of the day. Given the irregularities of attendance at school, and the restrictions on our own schedules, each child had about 2-4 sessions per week with us; our total length of work at the school was approximately three-fourths of the school year. Each day's session with a child lasted about 15-20 minutes, of which the first 5 or so were spent in the preliminaries, some 5-10 minutes on the "number tasks", and 5-10 in the closing play period. Although we made up our own instructional materials as we went along, and often extemporized on play materials and games, the pre-K teachers and the school principal did volunteer to provide some play objects of the sort found in the pre-K and K classrooms as usually described.

Because we did not wish to bias our data in the teachers' (or anyone else's) eyes we had planned originally to use only six children, one boy and one girl from each of the classes, but a seventh child, a girl, was added early in the project at the special request of one of the teachers who thought the added attention would be "good" for the child; and, in fact, the child quickly adopted us and insisted on having her turn with us. Since neither of us was fluent in Spanish, the teachers selected children who spoke English. At the opening of the project, we sat in and observed activities in each of the three classes for approximately two weeks, three times per week, and thus became familiar faces to the children. In our first two teaching sessions, we took the children to our working room in pairs to maintain their sense of security, but thereafter we called for them individually, and worked with them individually, without any evidence of timidity or withdrawal on any child's part. In fact, the children eagerly anticipated the sessions, and rarely hesitated to come to us. When they did hesitate, it was usually because of an especially intriguing or novel activity in their home classroom which they would have to cut short or forego if they came for their session with us. A number of children beyond those with whom we regularly worked would often try to become members of the group, calling out "take me, too", or "I want to go with you". A simple opening query to a scheduled subject as to "whether you would like to play some games with numbers today" routinely brought immediate assent, and the child would then be escorted to the day's working room by whichever one of us was on duty that day.

In our first sessions with the children, it was evident that each child: *a*) already knew some of the number names between "one" and "ten" ("zero" as a response was taught by us later); *b*) could recite at least some of the number names he knew in proper numerical sequence; *c*) could pick out visually, and point to, a few of the numbers in the range 1-10 when asked for them by name; *d*) could, on request, name some of the numbers when they were visually presented; *e*) would make some typical interconfusions of numbers, such as calling a "3" an "8", or vice versa; *f*) had learned to use his fingers at least occasionally to "count with", that is, to use his fingers as either a prompting device for rote recitation, or as an object set; *g*) could tell the number of objects when shown a set of two or three, and some a set of four. It was clear at the start of our project, therefore, that our subjects had already begun learning their numbers and "counting", but that basic sequences and fundamental tasks still remained to be thoroughly mastered.

We did all the teaching personally, without assistants. We carried our "lab" with us to the school every day in an attache case; we alternated days between us for the most part; we kept records and protocols of each child's performance on a daily basis; we met to exchange information after each day's work so that the man who was scheduled for the next day would know what had transpired the day before, and each of us kept in touch with the daily log and protocols of the other's sessions with the children. The resources we used to guide us in our initial thinking about mathematical behavior were sparse: *a*) some assumptions and reasoning primarily gained from studies of discrimination learning under laboratory conditions; *b*) official curricular and evaluations forms from the NYC Board of Education that had been devised for grades above pre-K; *c*) some teaching experience of one of us (W.N.S.) both in and outside the New York City public school system at several grade levels above pre-K; *d*) talks with the teachers of our pupil subjects; *e*) the available experimental literature on human "subitizing" and perception of numerosity; *f*) experiments on "counting" by animals; and the like. The total information derived from all these sources was small with regard to the practical instruction of young children. Moreover, the public school personnel, teachers and principal together, admitted they could be of very little help to us either in terms of their personal experience in such teaching, or in terms of educational literature they knew of on the topic. Because that background information is so fragmentary, we have not included a review of it here; it suffices to say that no alternative plan or attempt to organize experimental findings relevant to our teaching goals is to be found in the literature, and no strategy for reaching our goals has been suggested anywhere as an alternative to the one we arrived at as our first approach to this behavioral area.

On interview by us, it was clear that the teachers of young children cannot diagnose mathematical behavior. When they do identify a particular child's deficit in mathematical performance, they have little or no information about the discriminations required for a "correct" performance, nor about the interdependencies of the component discriminations that go into a "correct" performance, nor about how to correct a "wrong" performance. Moreover, they do not have reliable and precise techniques for measuring such discriminations as they can identify. By the same token, when a pupil shows strong mathematical talents, the teachers have no information about how such talents might have developed, save for broad guesses like "his mother works as a cashier". Two children that we observed illustrated these problems nicely. The first child had already learned to follow the teacher as a reinforcement source so closely that new discriminative stimuli had no chance of gaining control over his behavior. In task after task, this child would listen to the instructions, but would only respond by somewhat random guessing, until a sign from the teacher, such as a smile or small nod of the head, indicated that he had reached the correct answer; while guessing, this child would never look at the numbers or other materials that he was asked to identify, but would watch the teacher closely for a response to one of his guesses. We managed to overcome this pattern while the child was working with us, but the behavior still persisted in other contexts largely because his regular teacher found it difficult in the "home" room to break her own habit of responding to guesses, or to give prompt individual attention to this child so as to reinforce control by the numbers rather than by herself. The problem was exacerbated for this child in the "home" room whenever a standard class exercise used by the teacher had the children responding as a group to numbers or letters. From direct observation in this sort of situation, we could tell that many of the children, including our subject, had learned to respond not to the symbols presented by the teacher, but to cues from the teacher's person or from other members of the class, whom they watched consistently, and more or less covertly. The second child had a seemingly advanced number repertoire which exemplified a stimulus control problem of a different sort. She played games in which she labelled objects in monetary terms involving numbers. She would pick up objects and say things like "this costs five dollars", or "I have fifty cents and you have twenty cents". Although her repertoire of spoken numbers was well differentiated, none of the requisite discriminative cues appropriate to mathematical behavior had control of that verbal repertoire. Training the discriminations of these two children proved more difficult than with the other children because the irrelevant repertoire was reinstated periodically, thereby increasing the conflict with the correct repertoire each time a correct discrimination was accepted and reinforced either by us or anyone else.

It seems to us evident from many sources, including our observations and experiences with these children, that the performance called "counting" cannot be considered a unitary one. Rather, "counting" is a congeries of many different performances, varying from simple to complex. From any one of these, an observer can infer that a child is competent with numbers in sequence, that is, that the child can "count". The inference is based also upon the social context in which the child's performance is evaluated; for example, one number performance by a child may be acceptable at home, or in play situations with other children, and still not be sufficiently under appropriate stimulus control in the classroom to meet scholastic standards of "accuracy" (as in the case of the second child mentioned in the preceding paragraph).

We have probably not ourselves identified all the possible performances that are called "counting" in one context or another, nor have we analyzed completely those that we have identified. But we believe that we can now point to several behavior sequences that are included in the behavior category "counting", and that are involved in the later development of more complex mathematical behavior. We believe also that a more objective and less anecdotal analysis than the one we can give now of these early foundation behaviors can be undertaken with profit.

Provisionally, at least, "counting" can be broken down into the following distinguishable categories of performance in order of increasing complexity (after acclimatization and desensitization to numbers have taken place). It is around this breakdown that we are tentatively planning our teaching programs and research emphases, although we assuredly recognize that each category, particularly # 8, will need further specification when the detailed data begin to come in.

1. Learning the number names (for whatever range of numbers it is decided to use).
2. Rote recitation of number names in sequence.
3. Number recognition and identification.
   a) Visually presented numbers identified verbally; verbally presented numbers identified visually.
   b) Cross-modality matching of numbers and names; visual and auditory presentation; vision-touch; hearing-touch.
4. Responding in double sequences: "enumeration".
   a) Enumeration of similar objects, that is, using successive differentiated responses (e.g., verbal numbers in sequence) in correspondence with successive undifferentiated responses (e.g., picking up, or transferring, similar objects one by one).
   b) Matching "enumeration" to "instruction": counting in sequence, with the terminal response matched to, and designated as, the instructed "count".

5. Subitizing: differentiated number responses corresponding to an object set without the successive responses of "enumeration".
   a) Subitizing in several sense modalities.
   b) Cross-sensory transfer of subitizing.
6. Making numbers: motor performances in producing and in writing numbers.
7. Number concepts and sets: broadening "enumeration" to include heterogeneous classes of objects.
   a) Object sets.
   b) The "number of numbers".
8. Arithmetic: addition and subtraction.

The work we have done so far has been limited to the use of the numerals 1-9, with zero included both as a null set ("zero") and as an arithmetic position marker as in "10". As noted earlier, when our sessions with the children first began, we were able quickly to ascertain that they had already acquired rudimentary familiarity with visual and verbal numbers. thus, the first three stages of number performance shown above needed in the main only some strengthening and minor additions to the repertoire, rather than initial acquisition and differentiation of the performances. After four or five sessions with each child, we decided on a range of performances to be taught each child each day. New numbers were added slowly, day by day, until the child could quickly go through the sequence of all the elements 1-10.[4] At that stage we aimed for a performance battery consisting of abilities to: *a*) recite the numbers 1-10 in order, on command; *b*) follow our saying of each number name in sequence with pointing to the appropriate number when all ten were visually displayed in a scrambled arrangement; *c*) pick out and count an instructed number of objects from a group of similar ones; *d*) count the number of like objects displayed in a fixed configuration. All the children quickly developed their performances on the first two of these tasks: the rote recitation from 1 to 10 was the more easily perfected of the two, and only minor individual difficulties appeared in the learning to point correctly to the visually displayed number which we named.

At this point in the children's progress, our general ignorance of what mathematical behavior is really like was forcibly brought home when we tried to take the next step in our teaching. Innocently enough, we thought

[4] We recognize that we are not specifying in this report precisely how we brought our subjects along step by step in their number performances. The reason is that we are actually not certain of just what we were doing at each step to make the advance to the next step behaviorally possible. Whatever success we had in promoting our subjects' progress was the result of a mixture of our intuitive improvisations of teaching procedures with our occasionally more rational analyses of the behaviors we were confronted with. At this juncture, while anticipating a more experimental research approach in the future whereby we can quantitatively evaluate specific variables that govern "counting" behavior, we wish only to describe somewhat in field protocols the behavioral domain as it appears to us, some typical behavioral problems encountered therein, and some target behaviors that are possibly important teaching goals in that domain.

that step would prove an easy one. It was to combine performances (*a*) and (*b*) above, that is, when we had arranged the numbers 1-10 in a random visual pattern, we asked the child to point to the numbers in sequence himself while saying their names aloud. Not a single child was able to do that. The failure was both instructive and cautionary: the two performances had seemed to offer a logical and feasible next step in the learning by way of combination since they had already been mastered separately. Indeed, in many standard classroom teaching situations, it is undoubtedly assumed that once these component have been separately learned, the child can routinely or automatically initiate and complete the combined performance for himself. Children's difficulties with combining them might easily, in the standard school situation, be ascribed to "individual differences", or to failure of "intelligence", when in fact the difficulty would be found uniformly among all children who had not been specially taught to integrate these two performances. The unforeseen inability of our subjects to achieve that integration confirmed our belief that behavioral hurdles of unsuspected kinds may well exist when one or another single performance is accepted as the exemplar and index of mastery of all the complex behavior patterns that are thrown together in the category conventionally labeled "counting". But our experience has also convinced us that it should be possible, using both animals and children as subjects, to pinpoint more accurately than heretofore the conditions necessary for transfer or nontransfer from some mathematical performances to others, and to establish a progressive hierarchy of learning steps that would lead most efficiently to the varied performances a child must have in his repertoire if he is to exhibit proper "counting" in all its forms.

After some 20 or 30 sessions, our subjects had mastered the first three of the tasks outlined above, and showed no deterioration of the skills between sessions. Mastering the last task of counting objects displayed visually in fixed configurations, however, posed some new problems. Most of the difficulty arose with configurations of more than a few (say, two to four) elements. When a child pointed to, and named aloud, the objects in such a configuration in sequence, his fingers would often move in such spatial patterns that he would return to the same objects several times, advancing his spoken count each time without recognizing that some pointings were being repeated. If left uncorrected, some children could continue the repetition of pointings until they reached the limit of their verbal number repertoire. Such observations of the importance of visual configurations to "correct" counting naturally raise the question of what configurational properties aid or hinder the initial learning to count. Again, we believe that an experimental analysis, using both animal and child subjects, would lead to disclosure of the relevant variables and is a timely undertaking.

Once accomplished correctly, all the tasks in the provisional battery we developed at P.S. 101 were performed more and more rapidly. We began to find time in the short working sessions to add other tasks, and to investigate in a preliminary way some subtler variations of number manipulations in counting, and to later stages of number work such as the arithmetic of addition and subtraction.

Having learned to count accurately the number of elements in given geometric configurations (for example: ⁙ , ⁛ , ..., etc.), the children were able to transfer readily that counting or enumeration accuracy to a variety of other configurations, but not to others; and they could transfer to other objects in the same configurations they already knew, objects such as coins, pieces of chalk, pencils, etc. They were able also to transfer to groups of identical numbers (e.g., groups of 2s, or 7s, and so forth) if these were written in a straight line configuration on a blackboard while the child was watching. The major problem which emerged with these extensions was the maintenance of accuracy in counting. We tried to make certain that the child would be corrected immediately should he skip one of the elements as he counted them, or should he count one element more than once. The children were also able to count dissimilar objects comprising a visually presented set, so long as those objects were not numbers.

The latter observation led us to attempt some further tasks with numbers as the objects to be counted. A striking observation was that every one of our pupils failed at first when a group of numbers, each different and in random order, was displayed in a linear configuration on the blackboard, and the children asked to say "how many numbers" there were in the line. They failed at this even when the numbers were written slowly in a line before their eyes. Several children named the last number in the line, as if this were a possibly correct response to the instruction "how many numbers are there? ", although the numbers shown were in random sequence, and did not in any case contain all ten numerals. Some of the other children appeared to search for the largest number in the line, giving that as their reply. Only after their individual instruction over several sessions, and after being coached in pointing to successive elements in the line of numbers while counting aloud at each step, did consistently correct counting of the elements of a heterogeneous number set occur. Thus, the "number of numbers" presented a discontinuity in both the "concept", and in the routine performance, of "counting" and numerical sequencing, that had to be bridged by special training. In retrospect, the difficulties with this task might have been predicted as a forced outcome of the children's previous training. All the previous stages of learning had used visual numbers as discriminative stimuli trained to control differentiated verbal responses, i.e., the number "names". The effects of this training naturally persisted when the children were asked to perform a new and untrained complex discrimination in which previously discriminated members of a stimulus class had now to be generalized simply as members of another class, that of "number". Specifically, the instruction, "tell me how many numbers there are", probably produced two conflicting behaviors: the "how many" part of the instruction tapped behavior which had been trained only with objects other than numerals; the "numbers" in the query tapped behavior which had been trained only to produce the responses of number "names". The result was long hesitation before replying, and then a reply with a number "name", such as the last in the line (which comported with earlier learned sequential counting behavior), or the "name" of the largest number in the line (which had usually figured in the child's training as the last in a sequential enumeration performance). At the same time, the differential verbal repertoire controlled by the different visual numbers had to remain differentiated despite the fact that the new discrimination called for by the instruction would have to ignore that verbal repertoire if the reply to the instruction were to be correct. There can be no doubt that learning the so-called "concept of numbers as objects" presents an especially prominent, if predictable, problem in the development of complex mathematical behavior. Nor can the problem be easily sidestepped by modifying the preliminary stages of training. Both the differentiated responses and the stimulus control of those responses by different visual numbers are indispensable at the various stages of learning that precede the "number of numbers" stage. At the latter point, it may be that the difficulty encountered must be overcome with specialized training, rather than prevented by changing the early training of other enumeration performances. The problem certainly merits further attention, particularly with reference to the role of the "instructions" that are used to elicit the performance by a child, of counting numbers.

During the training stage of the "number of numbers" discrimination, it became clearer (to our way of thinking) that the counting of special sequences of numbers was a stage preliminary to arithmetic behavior. Among the special sequences were those which preserved the usual order of the numerical series, but which started at positions in the series other than the number 1. As the children improved in their enumeration of the "number of numbers", we tested some of the possible special sequences. At least one child learned, with only a small amount of practice, to count reliably the number of numbers in an ordered series beginning above 1; the others made progress with this task, but not nearly as fast or reliably as our star pupil. When all of them ad made some considerable progress in the task, we began to compound the counting of numbers into behavioral sequences that might be involved with elementary arithmetic. The first exercise consisted of advancing a count by 1 for an item newly added to the set. An important variable in this performance was the delay time between the termination of the initial counting and the presentation of the additional item. If it was added quickly after the initial counting which led into it was finished, then advancing the count by 1 was successful; when the delay was increased beyond some critical value (a different one for each child), the child often recounted the entire group, including the additional item, to reach the correct answer. The same problems arose when extending the count by more than 1; in this case, more than a few seconds' delay between the enumeration of the first group and the continuation of the count to include additional items often made the child recount the entire combined group. Only one child, the same one who had so rapidly mastered the counting of number groups wherein the numbers were in numerical order but did not begin with number 1, generalized this way of "adding" to its reverse of "subtraction". Not only could he extend the series from a previously completed count, and without special instruction reach the correct whole count, but he would, after some number of objects had been removed, volunteer the remainder. The boy literally invented subtraction in this way. It seems that his ability to arrive at a correct remainder was related to his ability to begin his counting with the number response "1" when the sequence of the numbers he was counting did not begin with 1. For him, the series was probably his own count rather than the list of numbers written by us on the blackboard, and his performance was all the more remarkable because the adding and subtracting of numbers as objects was surely more difficult a task than similar performances with ordinary objects. Our view and interpretation of his behavior, though plausible, could not be directly substantiated with this child. The difficulty of specifying the necessary and sufficient conditions for his performance is one reason for turning to analytic studies with animals that might give clues to the behavioral components that comprise the complex "counting" response chains required to combine ("add") or to separate ("subtract") groups of objects. These components are certainly different from the performances established through the classical "number tables" in which purely rote formulas become substitutes for addition and subtraction of number symbols by providing a mediating response chain to arrive at "correct answers" (as, "3 plus 4 equals 7"). Both types of performance would seem to have their uses in daily life (and perhaps even in the development of mathematicians!) but the differences between them, and perhaps their later simultaneous aplication in arithmetic behavior, call for analysis because they are important.

Near the end of the school year, we were able to introduce exercises in the "making" (writing) of numbers by the children All our subjects could hold a pencil, crayon, or piece of chalk to draw with Most of them could produce at least a few of the numbers 1-9 in recognizable form, though on occasion errors like mirror-writing (e.g., 3 or 7) would appear. In the main, the writing problems arose from awkward or improper starting points when drawing a number, since the children actually could make both curved and straight lines with their drawing tools. When given an outline figure, such as 8, 7, etc., to guide their drawing, almost all the children could keep their line within the figures (at least, when using the sizes of figures we did). When given incomplete figures to finish, such a 5, 7, the gaps were almost invariably bridged with a straight line even when that was inappropriate because the gap occurred in a curved portion of the figure. When we showed a correctly formed figure, however, the children's performances improved at once. It was certainly the case that none of the children was incapable of making the movements necessary to construct any number in the 0-9 series. Problems of writing numbers arise when the behavior components of complex writing have to be combined into a socially acceptable and aesthetic visual product. Little, if anything, is known about the motor performances involved here, and we may do some further work in this area at some future time. While visual presentation of a properly formed number as a model to copy may help in learning to write numbers, there are other procedures for training the stimulus discriminations and motor differentiations to guide children when they start the performance of writing complex numbers, or to guide the changes in hand direction when writing a number. At every stage of such motor teaching, among other considerations the matter of "step size" in shaping out the desired movements would be of constant concern.

Our plans for continuing work along the lines suggested above, if continuation is made possible by financial support from some agency, involve the following four major features, (1) There will be a change in site from P.S. 101 to several facilities closer to our home university base. This will cut down on our travel time, and we have fairly definite prospects of being assigned permanent space in which we can set up research operations of a more thorough sort than a despatch case. (2) The children available at our new sites will enable us to do what we could not in our earlier setting, that is, to work with children younger than the 3.5-4.5 year olds we have had heretofore. The new age range will be from early infancy to pre-K, and we should be able to explore performances that are preliminary to "counting" itself, as well as to those on which we have made the observations reported herein. (3) It is our aim to move past the types of protocol and field notes that we have been limited to in previous work, and to quantify our observations and data on mathematical behavior. This would require the installation of substantial technical equipment for stimulus presentation and for behavior recording, but we already have the designs for such apparatus partially worked out. (4) We have designed animal studies that key in with the child work at a number of stages. It is expected that the animal research program will expand as we learn more about mathematical behavior among children and about the possibilities of studying parallel processes among lower organisms. Given a broad and sustained attack upon the analysis of early mathematical behavior, we see no reason to doubt that it will be possible to develop teaching procedures that are reliable, efficient, and objectively defined to the point where regular teachers of such behavior in school situations may find them readily adoptable.