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EQUIVALENCE: A THEORETICAL OR A DESCRIPTIVE MODEL?

EQUIVALENCIA: ¿UN MODELO TEÓRICO O DESCRIPTIVO?

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ABSTRACT

The question is asked: Does the mathematical theory of sets provide a theoretical or a descriptive model of equivalence relations in behavior? The argument is made that mathematical set theory contains generalizations that encompass many real-world phenomena, including relations among the environmental and behavioral components of reinforcement contingencies. If pairs of elements that make up an n-term reinforcement contingency are members of an equivalence relation, then emergent relations between elements of the contingency simply illustrate the reflexive, symmetric, and transitive properties that define equivalence. Mathematical set theory defines the equivalence relation in a way that describes behavioral observations perfectly. The behavioral observations are therefore no more than exemplars of the mathematical generalization. Although the mathematical description of equivalence relations in behavior is characterized by features that a good theory must possess -consistency, coherency, productivity, and parsimony- no behavioral theory is involved in that description.

Key words: equivalence relations, set theory, contingency, theoretical vs. descriptive model.

RESUMEN

Se hace la pregunta: ¿provee la teoría matemática de conjuntos un modelo teórico o descriptivo de las relaciones de equivalencia en la conducta? Se hace el

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argumento de que la teoría matemática de conjuntos contiene generalizaciones que abarcan muchos fenómenos del mundo real, incluyendo relaciones entre los componentes ambientales y conductuales de las contingencias de reforzamiento. Si los pares de elementos que componen una contingencia de reforzamiento de n términos son miembros de una relación de equivalencia, entonces las relaciones emergentes entre los elementos de la contingencia simplemente ilustran las propiedades reflexiva, simétrica y transitiva que definen la equivalencia. La teoría matemática de conjuntos define la relación de equivalencia, de tal forma que describe perfectamente a las observaciones conductuales. Las observaciones conductuales no son, por tanto, más que ejemplos de la generalización matemática. A pesar de que la descripción matemática de las relaciones de equivalencia en la conducta está caracterizada por atributos que debe poseer una buena teoría -consistencia, coherencia, productividad y parsimonia- ninguna teoría conductual está involucrada en tal descripción.

Palabras clave: relaciones de equivalencia, teoría de conjuntos, contingencia, modelo teórico vs. modelo descriptivo.

William N. Schoenfeld never tried to teach by "passing out information" or by "imparting knowledge". For him, teaching was a social interaction, a two-stage process: first, listen to the student (or other speaker); then, ask questions that require the speaker to consider his/her words more carefully. And so, when I sent him my book on equivalence relations (Sidman, 1994), I waited eagerly for his questions. I knew they would teach me something new, that they would make me think differently than before. Indeed, after some wonderfully warming compliments about the contents of the book (this time, of course, reading rather than listening), he went on as I expected: "This is not to say that I don't have questions about your experimental work, nor theoretical questions about it." Unfortunately, however, he stopped there, and probably because of his failing health, did not respond to my request for more details. Nevertheless, just knowing that he had questions made me look for areas in the book that might require elaboration or perhaps even major changes. I found several, but will outline just one of those here. I can only hope it is one of the points that Schoenfeld would have wanted to see elaborated.

For many years now, my colleagues and I have been doing research in an area I call "equivalence relations and behavior", and I have come to talk about that research in a particular way. Here is the problem I want to highlight: Is my way of talking about what I have seen in my own laboratory and elsewhere a theory? Or is it a way of describing the observations *-and nothing more*? I have not seen much explicit discussion of a distinction between theoretical and descriptive models, and this, I am certain, is a topic that Schoenfeld could have helped me clarify.

Although no inclusive theoretical formulation has guided my research on equivalence relations, the absence of such guidance did not come about because I had anything against theory in principle. Any consistent, coherent, productive, and parsimonious way of defining and talking about empirical phenomena would find me receptive (Figure 1). I believe that those features make a theory worthwhile.

CONSISTENCY

COHERENCE

PRODUCTIVITY

PARSIMONY

Figure 1. The characteristics of a useful model.

In the course of our research, my colleagues and I eventually arrived at a way of talking about some of the things we saw in our laboratories -and in extensions to the world outside- that did seem consistent, coherent, productive, and parsimonious. Perhaps because of these characteristics, others in the field have called our kind of talk a theory or theoretical model. I am not sure, however, that such a characterization accurately describes what we have come up with. I do not believe that what we have proposed is a theoretical model. Rather, I believe it is a descriptive model, a useful framework within which to describe our data. Furthermore, because one can view the descriptive framework as a specific instance of an established mathematical generalization, I believe it is a kind of descriptive model that behavior analysts might find more widely useful.

Before presenting some background about the work that gave rise to the question of a distinction between two kinds of models, theoretical or descriptive, I want to emphasize that the current model was not something I came up with first and then tested via experiments. Although the experiments began for reasons that I still think are relevant -I was concerned about the origins of reading comprehension (Sidman, 1971)- I did not have the slightest notion of where the work was eventually going to take me. In the early studies, I even used the term, "equivalence", without realizing that I did not know how to define it. Whether the current model turns out to represent a theoretical or a descriptive formulation, then, my thinking has been inductive all the way.

The Basic Experiment

Let me first describe a set of procedures and findings that illustrate how I was led to my current way of talking about what we were seeing. These were some of the early experiments, done with five- to six-year-old children -probably brighter than average- and some older boys with such severe mental retardation that I usually also call them children. Figure 2 summarizes the procedures and the general experimental strategy.

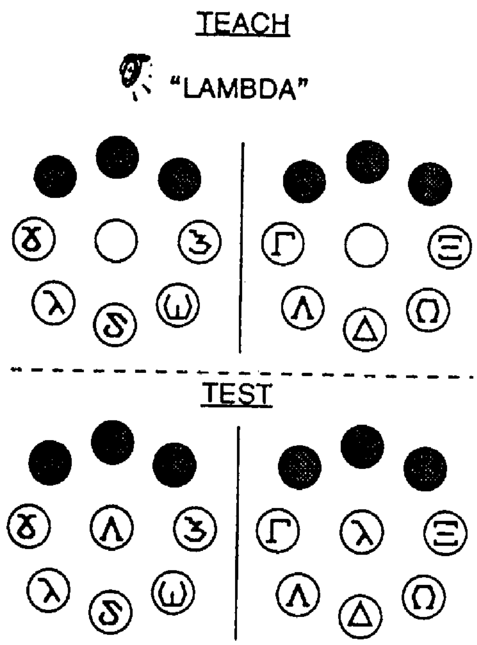

Figure 2. Diagrammatic examples of stimulus displays in the basic experiment. See text for procedural details.

A computer program draws visual stimuli -in this instance, forms that I shall call Greek letters- onto a matrix of nine windows, or keys (here, the uppermost three windows of the matrix are not used; they remain dark). The computer also presents prerecorded dictated words -in this instance, Greek letter names. Each teaching or test trial begins with a *sample* stimulus that is either auditory, as in the two top diagrams, or visual -always in the center window, as in the two bottom diagrams. When the sample is auditory, the center window remains illuminated but blank. Forms in the outer windows of the matrix are the *comparison* stimuli. The child's task is to compare these to the sample and to select one of the comparisons by touching it. If the child touches a comparison that the experimenters have decided is correct for that sample, the apparatus delivers a reinforcer -a token, a penny, a candy, or whatever we have found will keep the child's behavior going for a reasonable time.

In the top left-hand diagram, a trial begins with the dictated sample word, "lambda". The subject must select one of the Greek lower-case comparisons. If the child selects lambda, the computer delivers a reinforcer before presenting a different word to start the next trial; if the child selects something other than lambda -an incorrect choice- the only consequence is the start of the next trial. The dictated sample word varies from trial to trial and the positions of the comparison stimuli on the matrix also vary.

This procedure produces *conditional discriminations*; the subject's selection of a comparison stimulus on any trial becomes conditional on the particular sample for that trial. Although the subject's performance on tasks like this is often called *matching to sample*, the equivalence model I am leading up to forces me to define matching to sample in a way that differentiates it from conditional discrimination. For me, matching to sample involves equivalence relations, which possess properties that conditional relations do not require (Sidman *et. al*, 1982; Sidman & Tailby, 1982). I therefore use the term "conditional discrimination" rather than "matching to sample" as a name for the experimental procedure in our basic experiment.

Besides learning to match a dictated name to each of the lower-case letters, as in the top left-hand diagram, the child also learns to match those same dictated sample names to upper-case comparison letters -as in the top right-hand diagram. The same kinds of reinforcement contingencies prevail.

The subject then receives some tests. We call them *tests* for two reasons. First, the subject has to do something new. Having learned to match dictated letter names to lower-and upper-case letters (as in the top diagrams), she/he is now faced with the unfamiliar task of matching visual upper-case samples to lower-case comparisons (in the bottom left-hand diagram, the sample now appears in the center window), and of matching lower-case samples to upper-case comparisons (the bottom right-hand diagram). On test trials, there is no auditory stimulus. The second reason for calling these *tests* is that the child goes from trial to trial without receiving any indication of whether the choices are correct or incorrect. During tests, the apparatus provides no differential consequences.

This is where the real excitement occurs. Even though they had had no experience with these tasks, most subjects breeze through the tests; they match upper- to lower-case Greek letters, and lower- to upper-case, with few or no errors.

The same kind of thing happens with other kinds of stimuli. For example, with subjects who have not learned to read, one can substitute dictated picture names for the dictated letter names, printed picture names for the lower-case letters, and pictures for the upper-case letters. Then, one will find the subjects able to match the printed words to the pictures, and the pictures to the printed words. That is to say, they will be doing simple reading with comprehension, even without having been explicitly taught to do so.

An Expansion of the Basic Experiment

What seems to happen in these experiments is the establishment of several three-member equivalence classes. In this example, each class contains an auditory letter name, an upper-case letter, and a lower-case letter. What might be called the LAMBDA class consists of the auditory name "LAMBDA" along with upper- and lower-case lambda; other classes, each with three elements, might be called the GAMMA class, the XI class, the OMEGA class and the DELTA class.

Before getting into the equivalence model, it will be useful to outline an expansion of the basic experiment (Figure 3). Then, the results will serve as a basis for evaluating the utility of the equivalence model. I did this experiment because I wondered whether the conditional-discrimination procedure could generate classes that contained more than three elements.

As represented by AB and AC in the top diagram, the children learned two sets of conditional discriminations, matching dictated Greek letter names to upper- and to lower-case Greek letters. By convention, arrows always point from sample to comparison stimuli. On any trial, only one of the possible samples is presented, along with all three comparisons. Every trial, therefore, has one of the auditory A-stimuli as a sample, and the visual B- or C-stimuli as comparisons.

The same subjects also learned another two sets of conditional discriminations -DE and DF, as in the bottom diagram. On every trial, one of the English script letters (D-stimuli) was the sample, and three upper- or lower-case Greek letters (E or F) were comparisons. In these DE and DF conditional discriminations, all stimuli, including the samples, were visual.

At the end of this phase, subjects had learned four sets of conditional discriminations: Given either upper- or lower-case delta, sigma, and xi as comparisons (B or C in the top diagram), their choice on any trial became conditional on the dictated sample word; also, given either upper- or lower-case lambda, omega, and gamma as comparisons (E or F in the bottom diagram), their choice on any trial became conditional on the English script sample. We call these the *baseline* conditional discriminations, the ones we teach the subjects directly via explicit reinforcement contingencies.

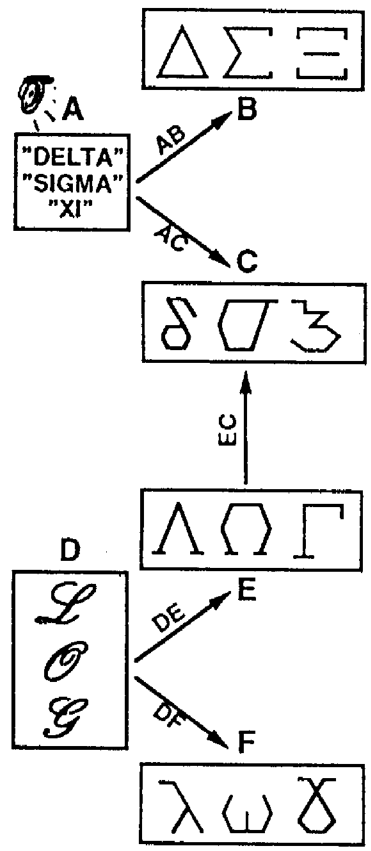

Figure 3. An expansion of the basic experiment. Arrows pointing from sample to comparison stimuli indicate baseline conditional discriminations.

These baseline conditional discriminations could have given rise to two sets of three-member classes: three ABC classes and three DEF classes. For example, one of the ABC classes would contain auditory "delta", upper-case delta, and lower-case delta; the other ABC classes would contain either the three "sigmas" or the three "xis". Similarly, each of the three DEF classes would contain a script letter ($\mathscr{L}$, $\mathscr{O}$, or $\mathscr{G}$) and the conditionally related upper- and lower-case lambda, omega, or gamma.

The two sets of classes -ABC and DEF- would be unrelated. Could we now expand these classes by merging them? To find out, we added the EC conditional discriminations to the baseline. In the EC conditional discriminations, subjects matched samples from the E-stimuli to comparisons from the C-stimuli. After subjects had learned EC, their choice of a C-stimulus (lower-case delta, sigma, or xi) on any trial depended on the particular E-stimulus (upper-case lambda, omega, or gamma) that was the sample.

The experimental question was this: Given that the original baseline had established equivalence classes, would teaching a subject to match a sample taken from one class to a comparison taken from a previously unrelated class bring all pairs of stimuli from those two classes into the same equivalence relation? We started with six three-member classes (three in the top diagram and three in the bottom); would the original six three-member classes merge into three six-member classes?

If each pair of three-member classes merged into a single six-member class, every stimulus in what we might call the DELTA class would become related to every stimulus in the $\mathscr{L}$ class; all SIGMA-class stimuli would become related to all of the $\mathscr{O}$-stimuli; and the same for members of the XI and $\mathscr{G}$ classes. To find out whether this class merger took place, we tested the subjects for all of the conditional discriminations that would be expected from three six-member classes. Figure 4 summarizes what happened.

An incredible explosion took place, behaviorally complex but, as we shall see, easy to describe. In this diagram, solid arrows again denote the directly taught baseline conditional discriminations (AB, AC, DE, DF, and EC). The dashed arrows denote conditional discriminations that emerged during tests -without differential reinforcement. Many subjects proved capable of all of the tested emergent conditional discriminations that were to be expected if the classes had merged. (With this particular matching-to-sample technique, tests with auditory words as comparisons could not be given because those comparisons, all dictated at the same time, would not have been discriminable).

The most complex instance was the BF test: given samples from the B-stimuli, and the three F-stimuli as comparisons, subjects selected lower-case lambda whenever upper-case delta was the sample, lower-case omega when upper-case sigma was the sample, and lower-case gamma with upper-case xi as the sample.

All in all, because each arrow denotes three conditional discriminations, subjects first learned 15 conditional discriminations directly (the five solid arrows), and then went on to demonstrate 60 new conditional discriminations

(20 dashed arrows) that they had never been explicitly taught. As we shall see, this remarkable productivity can be described parsimoniously as a behavioral example of the mathematical abstraction *equivalence relation*.

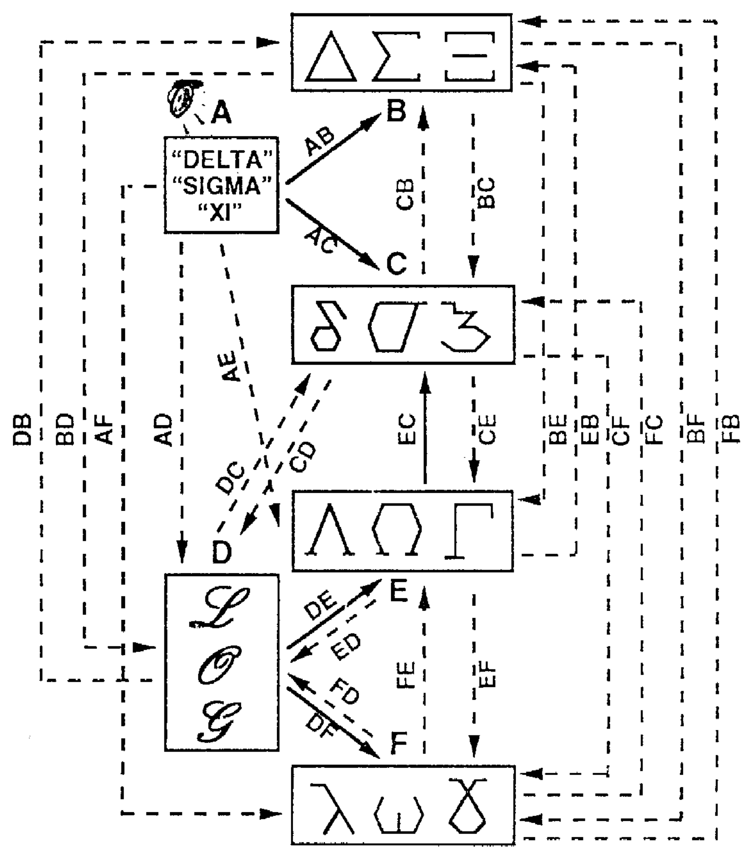

Figure 4. Solid arrows denote baseline conditional discriminations, as in Figure 3. Dashed arrows denote emergent conditional discriminations.

Equivalence Relations

After many more experiments and much discussion with students and other colleagues, it finally dawned on me that what I was seeing in the laboratory were specific examples of an already existing generalization. In that generalization, mathematical set theory defines the equivalence relation in a way that fits our observations perfectly. Mathematical abstractions are formulated without reference to real-world specifics, but they are often found to encompass many such specifics. Whitehead (1925, p.33) cited a simple example as follows: "...It is a general abstract truth of pure mathematics that any group of forty entities can be subdivided into two groups of twenty entities. We are therefore justified in concluding that a particular group of apples which we believe to contain forty members can be subdivided into two groups of apples of which each contains twenty members."

Although the mathematical generalization says nothing about apples or any other specific items, it turns out to apply to all. Similarly, the mathematical definition of the equivalence relation appears to possess tremendous generality: "Equivalence relations are found not only in every corner of mathematics, but in almost all the sciences" (Gellert, *et al.*, 1977). Our experiments have shown me that set theory's definition of the equivalence relation is a mathematical abstraction that also encompasses relations among the environmental and behavioral components of reinforcement contingencies. That is what I want to show next.

The Definition of Equivalence

Given that the explicitly taught conditional relations in the baseline (Figure 4) are also equivalence relations, the very definition of equivalence relations will be seen to *require* the emergence of the new conditional discriminations. If the solid arrows in Figure 4 denote concrete instances of the mathematical abstraction, *equivalence relation*, and if all of the related pairs represented by the solid arrows belong to the same equivalence relation, then all of the conditional discriminations represented by the dashed arrows must emerge. In our earliest experiments, I thought we were making discoveries when we saw these emergent conditional discriminations, but the definition of equivalence relations that we eventually arrived at turned those seeming discoveries into inevitabilities.

The proposition that the phenomena depicted in this complex diagram fit the mathematical definition of the equivalence relation is demonstrable. In order to clarify the relevance of the definition to the test results, let us get away from specifics like Greek letters and stimulus modalities (auditory or visual) and let us use a more general alpha-numeric designation of the baseline conditional discriminations (Figure 5).

Solid arrows still designate baseline conditional discriminations. The notation permits us to break out of the confines imposed by specific stimuli and thereby to avoid some questions that, although interesting, are not relevant to my present purpose. Note that the number of arrows has decreased; where a pair of dashed arrows had indicated that the same set of stimuli could serve either as samples or comparisons, I now use a single two-headed arrow. Also, because the A-stimuli are not limited to the auditory modality, I have added the emergent conditional discriminations in which A-stimuli would be the comparisons -BA, CA, DA, EA, and FA.

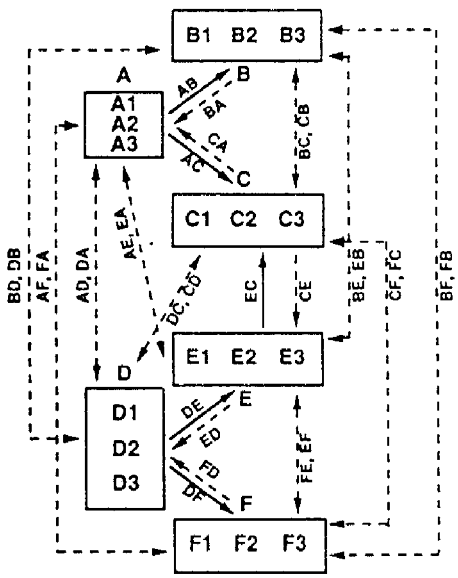

Figure 5. Like Figure 4, but with alpha-numeric designations for the stimuli, and additional emergent conditional discriminations in which the A-stimuli are comparisons. An arrow pointing in two directions indicates that each set of stimuli can serve either as samples or comparision.

Now, what are the defining features of the mathematical equivalence relation? It turns out that any of our baseline conditional discriminations that is also to fit the set-theory definition of an equivalence relation must possess three features: reflexivity, symmetry, and transitivity.

Reflexivity. If the conditionally related stimulus pairs in the baseline that Figure 5 summarizes -AB, AC, DE, DF, and EC- are also members of an equivalence relation, then reflexivity will bring the stimulus pairs denoted by AA, BB, CC, DD, EE, and FF into the relation. That is to say, the same conditional relation that holds between unlike stimuli in the baseline will, when tested, also hold between each of those stimuli and itself (Figure 6). For example, when given a test trial with Stimulus B1 as the sample, and Stimuli

B1, B2, and B3 as comparisons, a subject will select Comparison B1 even though the B1B1 conditional discrimination was never explicitly taught. The same goes for all of the other stimuli; given any one as a sample, subjects will relate that same stimulus to itself from among the comparisons. By testing for identity matching, therefore, we are also testing for reflexivity.

REFLEXIVITY

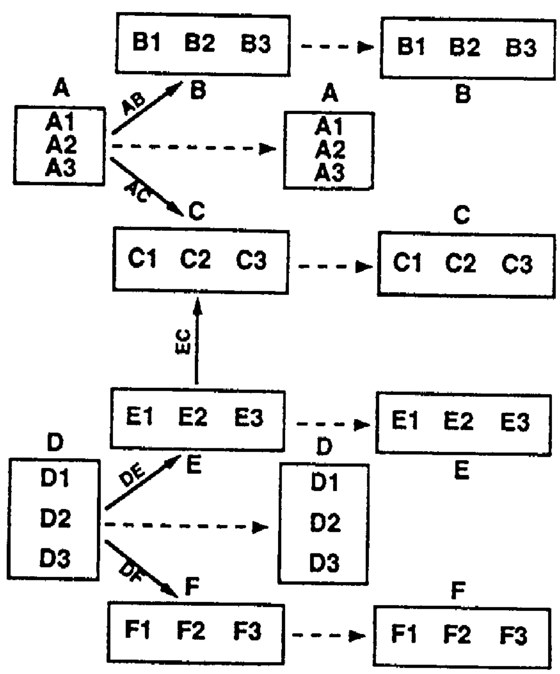

Figure 6. Emergent conditional discriminations (denoted by the dashed arrows) that confirm the reflexive property of the baseline relations.

Because we did not yet understand the mathematical definition of equivalence, we did not carry out definitive reflexivity tests in the early experiments. Other analyses, however, have confirmed their validity (for example, Carrigan & Sidman, 1992; Johnson & Sidman, 1993). The point I want to emphasize (and shall reiterate several times) is this: If the baseline conditional relations satisfy the definition of the equivalence relation, then the results of the identity matching tests are predictable. They are not predictable on the basis of any behavioral laws but are simply a consequence of the definition of equivalence.

Symmetry. The second required property of an equivalence relation is symmetry. If the conditionally related stimulus pairs in the baseline are members of an equivalence relation, then the symmetric versions of those pairs must also be members of the relation. That is to say, the sample-comparison pairs that are denoted by BA, CA, ED, FD, and CE will also be included in the relation (Figure 7). For example, when given a test trial with B1 as a sample, and A1, A2, and A3 as comparisons, a subject will select A1 even though the B1A1 conditional discrimination was never explicitly taught. The same goes for all of the other baseline conditional discriminations; given any former comparison as a sample, and the former samples as comparisons, subjects will relate the same two stimuli they had related in the baseline. The test for symmetry, then, is to reverse the original sample and comparison functions of the conditionally related baseline stimuli.

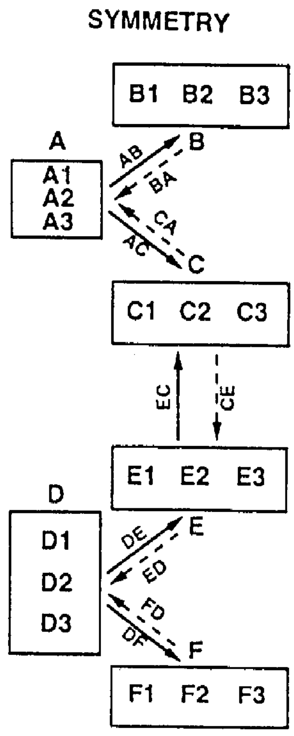

Figure 7. Emergent conditional discriminations (denoted by the dashed arrows) that confirm the symmetric property of the baseline relations.

In the experiment I have been describing, subjects performed with flying colors on all of the symmetric versions of the baseline conditional discriminations. Again, the point I want to emphasize is this: these emergent performances are predictable simply on the basis of the definition of the equivalence relation. Given that the baseline conditional relations are also equivalence relations, they must satisfy the symmetry requirement; the results of the symmetry tests become simply a consequence of the definition of equivalence.

Transitivity. The third required property of an equivalence relation is transitivity. A general illustration of transitivity is this: if x is related to y, and y is similarly related to z, then x will be bear that same relation to z. As an example, let us first look at the two sets of baseline conditional discriminations that Figure 8 summarizes as DE and EC. If the related DE and EC pairs are members of an equivalence relation, the conditional relation that holds between Sample D1 and Comparison E1, and between Sample E1 and Comparison C1, will also hold between Sample D1 and Comparison C1, even though the D1C1 conditional discrimination was never explicitly taught. The same goes for all of the other pairs of baseline relations that contain an element in common.

A way of testing for transitivity, then, is to start with two sets of baseline conditional discriminations in which the comparisons from one set serve also as samples in the other. Then, in the tests, ask the subject to relate samples from the first baseline set to comparisons from the second.

In this diagram, all of the emergent performances that the dashed arrows represent are predictable simply on the basis of our definition of the equivalence relation. Indeed, our subjects proved capable of all of the tested conditional discriminations, even those complex ones that required more than just two pairs of related items with a common element. As before, then, given that the baseline conditional relations are also equivalence relations, they must satisfy the transitivity requirement; as with the reflexivity and symmetry tests, the results of the transitivity tests become simply a consequence of the definition of equivalence.

Theoretical or Descriptive Model?

Figure 4 showed the explicitly taught baseline and all of the emergent performances in our original experiment. Having applied the set theory definition of the equivalence relation to our behavioral observations (Figures 6, 7, and 8), we now find that the emergent conditional discriminations in our original experiment are no longer mysterious. All those performances are to be expected if the baseline contingencies have established an equivalence relation.

If all of the conditionally related stimulus pairs in the baseline are members of an equivalence relation, then the emergent performances simply illustrate that relation's required properties -reflexivity, symmetry, and transitivity.

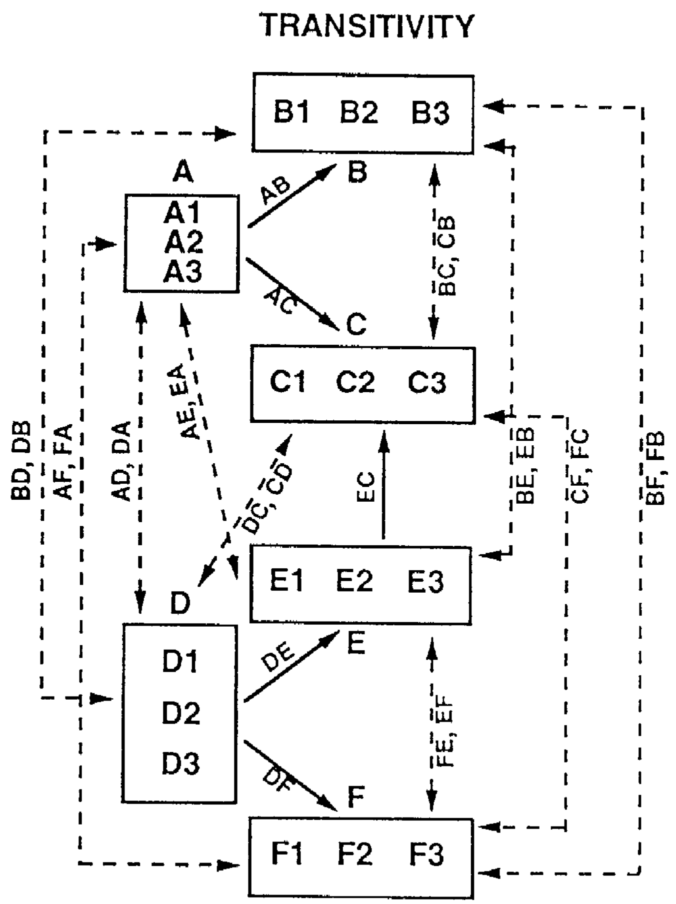

Figure 8. Emergent conditional discriminations (denoted by the dashed arrows) that confirm the transitive property of the baseline relations.

Given an equivalence relation, then, its very definition tells us that all of the stimulus pairs that are represented by dashed arrows in Figure 5 are predictable. If even one of those pairs failed to emerge, we would have to conclude that the mathematical abstraction, equivalence relation, does not describe our data. Such a failure would not disprove the equivalence model;

there is no way our behavioral data could either prove or disprove the mathematical model of the equivalence relation.

One assumption here is, of course, clearly theoretical: namely, the hypothesis that our behavioral data represent specific instances of the mathematical abstraction, *equivalence relation*. Unlike most psychological theories, however, this simple hypothesis assumes no unobserved or unobservable structures, entities, or processes. It just places our behavioral observations within a specified descriptive framework. If the assumption is correct, then the simple statement that we are dealing with an equivalence relation provides a shorthand description of all the data in Figure 9. We find ourselves with a consistent, coherent, productive, and parsimonious description of our data.

THE EQUIVALENCE RELATION

BASELINE:

aRb, aRc, dRe, dRf, eRc

by Reflexivity:

aRa, bRb, cRc, dRd, eRe, fRf

by Symmetry:

bRa, cRa, cRe, eRd, fRd

by Transitivity:

aRd, aRe, aRf, bRc, bRd, bRe, bRf, cRb, cRd, cRf
dRa, dRb, dRc, eRa, eRb, eRf, fRa, fRb, fRc, fRe

Figure 9. All of the stimulus pairs that comprise the equivalence relation, R.

Here is a shorthand way of illustrating these features of the descriptive system. The letter "R" means "bears a particular relation to". "aRb" means that a bears the relation to b; "aRa" means that a bears that same relation to itself;

and so on. We start with our explicitly taught baseline and assume that each of the indicated pairs is a member of the same equivalence relation, R. (In our experiment, each expression actually represents three pairs of related stimuli). Then, by the defined properties of reflexivity, symmetry, and transitivity, all of the other listed pairs must also be members of that equivalence relation.

Consistency. The consistency of our description lies in its replicability. It has been validated in many laboratories and classrooms, with many different kinds of stimuli, with many varieties of subjects, with varying numbers of equivalence classes, with varying class sizes, and with several different teaching and testing procedures.

Coherence. Criteria for coherence are complex, not to be thoroughly explored here. The coherence of our descriptive system lies first in its internal cohesiveness. All of the baseline and emergent conditional relations stand or fall together as examples or nonexamples of the same generalization. If a conditional discrimination is reflexive but not symmetric, the pairs of related stimuli cannot be described as belonging to an equivalence relation; if a conditional discrimination is reflexive and symmetric but not transitive, the pairs of related stimuli do not constitute elements of an equivalence relation; a baseline in which the ab and ac conditional discriminations belong to different equivalence relations (aR_1b and aR_2c; see, for example, Steele and Hayes, 1991) will require a different way of describing the test results.

Another aspect of the descriptive system's coherence is its compatibility with other aspects of mathematical set theory. Along with the equivalence relation, other concepts in set theory also prove useful in describing many of our findings. Among these are the abstractions, *set union* and *set intersection*. It is rare, probably even impossible, for any element to belong to just one class. When two or more classes have members in common, they may merge (set union) or remain independent (set intersection). Contextual factors determine whether set union or intersection takes place.

Let us look at just one of many possible illustrations. Start with a conditional discrimination in which the sample/comparison pairs are members of an equivalence relation. If one of the comparisons is then made to serve as the positive stimulus in a three-term contingency -a simple discrimination- the sample that was conditionally related to that comparison will also be found to function as a positive discriminative stimulus (de Rose, McIlvane, Dube, Galpin, & Stoddard, 1988; de Rose, McIlvane, Dube, & Stoddard, 1988). It turns out, therefore, that phenomena we have traditionally attributed to a behavioral process called "transfer of function" can be described more parsimoniously as the merger of classes that possess elements in common -an example of class union.

Like set-theory's definition of equivalence, its definition of class union turns seeming discoveries into confirmable inevitabilities. Although the behavioral observation here is the display of a new function by the sample stimulus, we need not bring in a separate behavioral process to account for it. Given the applicability of the mathematical abstractions to behavioral specifics, any demonstration of emergent duality of function by a particular stimulus takes its place within the descriptive system.

Productivity. One obvious kind of productivity comes from the predictability of the emergent conditional discriminations. A second kind of productivity is represented by the sheer number of new conditional discriminations that may emerge; the only limit to the number of emergent relations is the number of elements in the baseline itself. A third kind of productivity is illustrated by the term, *emerge*; the emergent relations are additions to the subject's preexisting behavioral repertoire. A related spinoff has been the use of the equivalence model in the planning of effective instructional sequences.

And a fourth kind of productivity is indicated by the new questions that have arisen about the possible effects of procedural variations, about relevance to other basic and applied areas of behavior analysis, and about the place of this descriptive system within the broader behavior analytic formulation. The literature is too vast even to begin to cite it. This fourth aspect of productivity is perhaps also to be categorized under *coherence*.

Parsimony. The parsimony is obvious. Attributing membership in an equivalence class to each of the related stimulus pairs in the baseline permits us then to specify all of the stimulus pairs that must also be components of the class. All of the emergent conditional discriminations are immediately described under the umbrella of a single abstraction: the equivalence relation.

We find, then, that our system possesses qualities that make a theory useful: consistency, coherence, productivity, and parsimony. Does that mean I have proposed a theory? I think not. The theory, which says nothing about behavior, was there already, waiting to be found. Built into set theory's definition of the equivalence relation are a large number of regularities. Those regularities define the equivalence relation. Any relation that is also to be classified as an equivalence relation must exhibit those regularities. The mathematically stated regularities do not explain there markable explosion of performances that Figure 5 summarizes. They do, however, provide a descriptive framework into which that explosion fits.

The necessity of the emergent performances gives the descriptive system one of the most often cited characteristics of a good explanatory theory; it permits us to make predictions. The predictions, however, do not come from any behavioral theory; they are already incorporated in the *mathematical* definition of the equivalence relation. The behavior-analytic task is to find out whether the regularities the descriptive system requires are in fact observed. If a subject proves incapable of any of the tested performances, we would have to conclude not that the equivalence model is wrong but simply that it does not describe these data. And if a subject does produce all of the tested performances, we have to conclude not that the equivalence model is supported but simply that it does describe these data. That the kinds of regularities we observed in our early experiments have been so reliably confirmed continues to astonish me. I know of no formal behavioral theory that makes so many behavioral observations so consistently, coherently, and parsimoniously predictable.

It is probably important for me to reiterate here, as I have done elsewhere (Sidman, 1994), that I do not use *equivalence relation* to refer either to a theoretical entity or to processes or entities that are beyond observation. For me, the term simply summarizes a set of observed regularities. Because clarity of communication often makes certain terms of common speech useful, I have sometimes referred to an equivalence relation as if our procedures had established, formed, or created it. The equivalence relation, however, is not actually established, formed, or created; it does not exist, either in reality or as a theoretical construct. It is defined solely by the predictable emergence of new analytic units of behavior from previously demonstrated units (Sidman, 1986).

And so, the set of ordered pairs that defines whether or not any particular relation is also an equivalence relation does not constitute a theoretical entity. Although one does not always observe the defining pairs of events, these are always potentially observable by means of conditional-discrimination tests for the properties that define an equivalence relation.

We find, then, that the mathematical theory of sets seems to agree closely with behavioral reality. If there is any discovery here, that is it. To have confirmed this correspondence means simply that the science of behavior has something in common with the natural sciences.

One consequence, however, that some have found disturbing, is the introduction of a new set of terms, which we might call the "language of equivalence". I, too, take a conservative approach to new terminology. If existing terms can serve the purpose, a new terminology is unnecessary and the labor required to master it is wasted energy. But the language of equivalence provides a descriptive framework that preexisting technical formulations do not make available. I do not believe that any set of classical behavior-analytic terms encompasses the results of the experiments I have described.

On the other hand, the language of equivalence does not preempt any classical terms or concepts; instead, it extends some of them. For example, the behavioral observations that define an equivalence relation can be seen to fall into the *tact* and in to the *autoclitic* categories in Skinner's formulation of verbal behavior. Far from dispensing with the concepts of tact and autoclitic, the equivalence relation can be seen to bring out previously unspecified features of those relations. The equivalence relation, for example, can be viewed as a rigorous substitute for the popular concept of *correspondence* between words and things, a concept which, as Skinner convincingly argued, is not enlightening (Skinner, 1957). The equivalence relation does not provide a theory of correspondence. What it does provide is an experimentally verifiable description of at least some of the phenomena that lead people to talk about correspondence. I believe that whenever people do talk about *meaning* as correspondence, equivalence relations are involved.

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