THE COMPLICATED PAIRING BETWEEN DYNAMIC SYSTEMS TECHNIQUES AND ECONOMICS

José Carlos Ramírez Sánchez
Centro de Regulación Energética y Economía del Desarrollo,
Universidad Panamericana (Mexico)
Corresponding author: jcramirezs@up.edu.mx
Adolfo García de la Sienra
Institute of Philosophy, Universidad Veracruzana (Mexico)

Manuscript received 9 June 2020; final version received 11 August 2020.

ABSTRACT

This paper aims to analyze the impact of the dynamic systems techniques (DST) on the recent development of neoclassical Economics. Through the use of a classification of research papers and two models, we study how mainstream economists translate concepts into dynamic formats. The main conclusions are: (*i*) DST have expanded knowledge in Economics by revealing new types of equilibria and tightening interrelationships among sub-disciplines; (*ii*) despite this undeniable success, some economists criticize how assumptions and concepts are reduced to technical expressions to ease their mathematical adaptation; and (*iii*) there is no neutral method to build dynamic models.

Keywords: Mathematization, non-traditional dynamic equilibria, the realism of assumptions, mathematical reductionism, Malthusian dynamics.
JEL Classification: B4, C6.

http://dx.doi.org/10.22201/fe.01851667p.2020.314.76042

EL COMPLICADO MARIDAJE ENTRE LAS TÉCNICAS DE LOS SISTEMAS DINÁMICOS Y ECONOMÍA

RESUMEN

Este documento analiza el impacto de las técnicas de los sistemas dinámicos (TSD) en el desarrollo reciente de la economía neoclásica. Para tal efecto, elaboramos una clasificación de artículos publicados en las principales revistas y dos modelos con el fin de estudiar la forma en que los economistas traducen los conceptos en formatos dinámicos. Las principales conclusiones son: (*i*) las TSD han expandido el conocimiento en economía mediante la fundamentación de nuevos equilibrios y el reforzamiento de las relaciones entre sus subdisciplinas; (*ii*) no obstante esta aportación, algunos economistas critican los procedimientos usados para reducir los supuestos y conceptos en expresiones técnicas que faciliten su adaptación matemática, y (*iii*) no hay un método neutral para construir modelos dinámicos.

Palabras clave: matematización, equilibrios dinámicos no tradicionales, realismo de supuestos, reduccionismo matemático, dinámica maltusiana.

Clasificación JEL: B4, C6.

1. INTRODUCTION

This paper is about the mathematization of Economics[1]. It aims to analyze the consequences of the gradual incorporation of the dynamic systems techniques (DST) in Economics during the last

[1] Hereinafter we will refer to the neoclassical Economics simply as Economics, unless otherwise indicated. Mathematization, axiomatization, and methodological formalization are elements of what Backhouse (1998) calls the process of formalization in Economics. While mathematization refers to the outcomes of the application of Mathematics in Economics, the other two elements indicate how such an application is carried out (Beed and Kane, 1991). In particular, axiomatization is the process of deriving propositions from a set of axioms attending well-defined logic rules, and methodological formalization is the utilization of mathematical methods in the solution of specific economic problems (Backhouse, 1998).

three decades[2]. In particular, the paper shows how mainstream economists adopt specific assumptions and mathematical tools to translate economic concepts into dynamic formats.

The choice of topic and time is relevant for at least two reasons. First, because the link between DST and Economics shapes the new mathematical orientation of the latter over the recent past (Weintraub, 2002). Through DST's extensive use, some branches of Economics have been able to expand their scope by studying properties of market mechanisms that are inherently unstable in their dynamics and not only deterministic and stable[3]. The second reason is that the future destiny of Mathematical Economics (ME) appears to be also linked to DST since, according to Holt, Rosser, and Colander (2011), the "era of complexity" is intended to replace the "neoclassical era" of ME. This new era, which is barely in its infancy, comprises the work of economists with approaches that assume interactions between heterogeneous agents, perpetual novelty, and dynamics without an optimal equilibrium (Arthur, Durlauf, and Lane, 1997). In other words, they are works that demand new ways of applying DST in Economics because their authors are reluctant to accept the neoclassical view of dynamics.

Both reasons make the study of the relation between DST and Economics a relevant issue to understand the price paid by the latter for having to conform to specific dynamic formats. The strong opposition of some economists to the particular way this pairing takes place is a warning that should be taken into account, mainly because any attempt at formalization involves the risk of leaving aside some concepts (mostly non-quantifiable) in the dynamic analysis. Which criteria do mainstream economists consider to pick up or discard determined elements? Why

---

[2] A dynamical system is a mathematical prescription about the way a non-empty set *X* evolves with time. It consists then of a set of state variables *X* that describes the position of a system at any time, and the dynamics or rule of change. DST is understood to mean all the mathematical techniques that deal with the specification of pairs of mathematical objects (*X* and dynamics) under the assumptions that they preserve a measure on the Euclidean space R (ergodic dynamic systems) or that *X* varies only continuously (topological dynamic systems) [see Bhatia and Szego, 2002; Hirsch, Smale, and Devaney, 2004].

[3] By adopting non-linear models, economists have tended to incorporate the whole spectrum of "motions," characterizing the dynamical behavior of any trajectory, namely: Steady-state, periodic, semi-periodic, and complex dynamics (Ott, 2002).

do they prefer to keep the mathematical structure unchanged rather than relaxing restrictive assumptions? We shall address these sorts of questions below.

The document has four additional sections. The second presents data from a classification of research papers to illustrate the consequences of the penetration of DST into specific fields of Economics. The following two sections explain these consequences through an analysis of the degree of realism of the assumptions and the conceptual validity of the dynamic models. In particular, the third section concentrates on how assumptions, like bounded rationality, are considered by mainstream and non-mainstream economists. The fourth develops two models to show how variations in dynamic techniques may lead to establishing different perspectives on the same economic problem. In the fifth section, conclusions summarize the main findings.

2. ADVANCES OF DST IN ECONOMICS

The start of the third and most recent phase of the mathematization in Economics occurred around the 1960s and its main feature is the turn of the ME literature towards dynamic analysis (Varian, 1991; Weintraub, 1991)[4]. Ramírez and Juárez (2009) find, in effect, that 24.48% of the 2,835 core-articles published between 1990 and 2004 in the most influential journals of the discipline (*American Economic Review*, *Econometrica*, *Journal of Political Economy* and *Journal of Economic Theory*), incorporated dynamic analysis. This result means that, in less than fifteen years, the percentage of articles with some dynamic content more than doubled that recorded between 1980 and 1990 in *the American Economic Review* and *Econometrica* journals (11%)[5].

---

4 The other two phases are characterized, first, by the development of Microeconomics (1838-1940) and, second (1940-1960), by the modern establishment of Macroeconomics, Microeconomics, and Growth Theory (Mirowski, 1989; Arrow and Intriligator, 1991).

5 By dynamic content, the authors refer to the use of any pair (*X* and dynamics) in the elaboration of a theoretical model or in the specification of a parametric model. We apply the same definition below. It is important to remark that the four surveyed journals are the most influential in Economics according to various score rankings (see Kailatzidakis, Mamuneas, and Stengos, 2011).

Among the reasons given to explain the rapid expansion of DST, there are two particularly illustrative. The first is that, unlike the early stages of economic models, economists have to cope now with non-linear mathematical specifications that require professional training in areas relatively foreign to Economics (such as Topology, Stochastic Optimization, and Differential Games; see Boldrin and Woodford, 1992). The flourishing of Econophysics during this period, together with its range of techniques derived from Thermodynamics, is an example of the new influence of these fields on different areas of Economics, particularly Finance (Bali, 2011).

The second reason is that DST have spread rapidly throughout most of Economics by venturing into areas typically dominated by static analysis. In this regard, Ramírez and Juárez (2009) indicate that between 1990 and 2004, articles in the 14 sub-disciplines classified as static, which had incorporated dynamic techniques, represented a little more than 50% of the sample; a surprising percentage considering that ten years earlier, the other two remaining naturally dynamic sub-disciplines, Macroeconomics and Economic Growth, accounted for 77% of the total. After updating the sample originally elaborated by Ramírez and Juárez (2009), we find that this proportion remained roughly the same for the period 2005-2010 (see Table 1)[6].

The rapid spread of DST has fostered connectivity between sub-disciplines. Authors now seek to combine various elements from different sub-disciplines not only to make models more realistic but also to explore the behavior of critical points under new constraints. Let us think, for instance, in the development of International Trade and Economic Growth. Both sub-disciplines recently merged again after having remained split for more than a century, as a result of a radical change in the long-standing view on international trade (Afonso, 2001). With the advent of the endogenous growth theory in the 1980s, some

[6] The classification is not free of arbitrariness because it is not so rigorous to estimate an empirical hypothesis as to test formal equilibria in a theoretical model. The purpose of the classification is purely heuristic, as it only seeks to show the change in the subject content of the sub-disciplines using any aspect of DST. The original sample by Ramírez and Juárez (2009) covers core-articles (2,835), notes and special issues for invitation (1,509). We only focus on the core-articles (1,163 of a total of 1,736).

Table 1. Number of core-articles with dynamic content published by the four leading mainstream journals, 1990-2010

Sample	Ramírez and Juárez	Updated sample
Sub-disciplines	**1990-2004**	**2005-2010**
Traditionally dynamic	335 (11.82%)	139 (11.95%)
Non-traditionally dynamic	359 (12.66%)	142 (12.20%)
Sample size	2,835	1,163

Note: Traditionally dynamic sub-disciplines are Macroeconomics and Economic Growth. Non-Traditionally dynamic sub-disciplines consist of all the others, mentioned in Table 2. These samples include only core-articles that represent 65% (1990-2004) and 67% (2005-2010) of the original sample size in both periods.
Source: Own elaboration.

economists started to realize that innovation was a fundamental part of international trade. In particular, economists began to consider new theoretical elements affecting the dynamics of innovation, like trade openness, geographic structure of international trade or capacity for internal technological adaptation, as drivers of human capital and, consequently, of economic growth (Keller, 2002). The introduction of these elements into economic growth models allowed economists to build up more realistic scenarios, but at the expense of complicating the calculation of equilibria that fitted the new concept of "open economy."

As a by-product of this connectivity, optimization problems have become progressively more complex. In addition to the typical saddle points, centers, spirals, or nodes, there are articles containing equilibria that differ from the two traditional types of motion: Steady state (when the system ceases its motion) and periodic (when the system enters limit cycles). Now it is common to find researchers dealing with quasi-periodic and complex dynamic motions in some parts of the long-term paths before reaching the definitive equilibrium (Day, 1983). The treatment of these new critical points demands more sophisticated motion equations (logistic maps or higher-order equations differential systems) as well as more complicated ways of optimizing sequential decision making over time and under uncertainty.

Table 2. Groups of core-articles with dynamic content published by the four leading mainstream journals, 1990-2010

Sample	Ramírez and Juárez	Updated sample
Groups	**1990-2004**	**2005-2010**
I	95	36
II	264	106
III	335	139

Note: Group I: Economic History, Environmental Economics, Welfare Economics, Industrial Economics, Political Economy, Regional Economics, Development Economics, and Institutional Economics. Group II: Game Theory, Labour Economics, Experimental Economics, Microeconomics, Finance, and International Trade. Group III. Macroeconomics and Economic Growth.
Source: Own elaboration.

However, it is worth noting that this technical sophistication is not the same throughout the literature, mainly because most articles treat the concept of dynamics differently. There are articles in which the aim is to test differential or difference equations econometrically but without considering any analytical or qualitative treatment of the trajectories. Examples of these articles are found in the first group of Table 2, which includes studies of Economic History, Welfare Economics, and another six sub-disciplines. There are also studies in the fields of Game Theory and Labor Economics, such as those in the second group of Table 2, that go beyond a simple econometric estimation and seek to calculate deterministic, stochastic equilibria based on evolutionary strategies for multiple stages games (Binmore, Piccione, and Samuelson, 1998). Lastly, there is a third group, the most numerous and heterogeneous of all, characterized by maintaining diverse positions. This group is composed, on the one hand, by authors in the purest neoclassical tradition who develop dynamic models favoring unique, stable equilibria (Howitt, 1999) and, on the other hand, by economists interested in predicting the existence of multiple, unstable equilibria under the same assumptions of traditional growth models (Backus, Kehoe, and Kydland, 1992). Between these poles, there is a sub-group, not as large but extremely representative of authors who recognize the inadequacy of traditional analysis due to the restrictive nature of the assumptions (see Mitra and Nishimura, 2001).

Despite these differences, it is clear that the incorporation of DST has significantly benefited the development of Economics. The availability of new DST has allowed some fields of Economics to flourish, such as the two-sex population theory, Econophysics, endogenous growth theory, or the new generation of matching models (Noldeke and Van Damme, 1990). Likewise, some practical problems linked to dynamic financial derivatives- pricing would be unthinkable without the support of dynamic stochastic optimization techniques. By using optimal control or stochastic dynamic programming techniques, authors are now capable of solving more theoretically-oriented problems of risk measures that would have been impossible to model utilizing the traditional tools of corporate finance. For these kinds of reasons, modern ME would be inconceivable without the support of DST, either for generating new ideas or for rejecting other long-accepted ones.

2.1. Alternative points of view

As it is common in different fields of knowledge, not all share the same optimism for new developments, especially in Economics, where the application of Mathematics is viewed with suspicion by many. One can realize this immediately as soon as one begins to review the works of economists who publish in less orthodox journals. In particular, two conflicting points stand out when comparing the journals of the sample with others.

The first point has to do with the concept of motion of an economy used by most macroeconomic models of the third group in Table 2. In such models, the word motion is deemed by non-mainstream economists as extraordinarily narrow and instead linked to the way physicists formalize inanimate physical entities. With that concept at work, Lorenz (2009) says that dynamic equilibria do not seem derived from a model, in the sense that they appear more as displacements from a fixed point than as a result of internal adjustments to the system. Dynamic models need to include feedback systems in which larger equation systems record the new information acquired by agents. Modeling the dynamics of an economy with fixed, deterministic laws of motion and under idealized conditions, is then seen as a contradiction in itself. According to Velupillai (2011), modern economies must be analyzed with the help of complex dynamic equilibria because they are the most complex of all dynamic systems.

The second point deals with the way neoclassical authors define the set of state variables *X*. As explained in Hirsch, Smale, and Devaney (2004), there must be a close correspondence between *X* and the rule of change in order to set the correct dynamics of any phenomenon under study; otherwise, its dynamical nature would be ill-specified. For this reason, it is wrong to combine, for instance, non-linear differential equations with state variables in which bifurcations are absent since the range of *X* would not be possible to identify.

In most articles of the four mainstream journals, the range of *X* considers only the values along the stable branch of a saddle point. The reason lies in neoclassical economists' stressed tendency to favor equilibria whose nature is asymptotically stable according to Lyapunov's criteria. They think of the remaining states or points on the *X* path over time as temporary disturbances near the steady-state. Hence, the introduction of systems of high-order differential equations, that supposedly intend to capture more complex dynamic behavior, should not always be seen as a genuine attempt to make the dynamic analysis more realistic but as a merely formal way to show sub-optimum unstable equilibria. Authors are more concerned with stable orbital equilibria rather than structural stability and so tend to use smooth differential dynamic systems that yield equilibria around an isolated critical point. The rules of change involving logistic maps or Duffing-like equations are regularly discarded because they produce tent maps, manifolds, or different kinds of chaos that prevent reaching stable critical points.

This peculiar correspondence between the space of state variables and the rules of change is not uncommon in Economics. It is a practice that is rooted in the axiomatic method, mainly fostered by Debreu (1984, 1991) and the particular methodological formalization adopted by mainstream economists. In both cases, the idea of unique and stable equilibria is omnipresent. In fact, without that concept of equilibrium, the use of mathematics in Economics would lose meaning, since otherwise the chain of reasoning of any model could no longer be broken down into is elementary steps. Mathematics helps the models of equilibrium make the chain of reasoning credible (Backhouse, 1998).

How does this breaking down process work? The most straightforward answer comes from the axiomatic method. According to Duppe (2010), this method presupposes the separation of economic content from mathematical reasoning because the formal structure does not require any interpretation. During the five critical steps of the process of axiomatization (selecting primitive concepts, representing these concepts as mathematical objects, specifying assumptions, deriving consequences, and interpreting), the only thing that matters is the mathematical structure. The fifth step, interpretation, is foreign to the first four ones because it is not considered an activity that belongs to the stages of logical rigor. It is instead a thing to be discovered (Boyland and O'Gorman, 2007; Duppe, 2010).

Each of the first four steps shaping the formal structure is subject to a rigorous deductive process free of logical contradictions. Neither stage is independent of the other. However, insofar as these steps are empty of economic content, the economist's task reduces thus to fill the formal structures by making their interpretations pass the "acid test." Interpretations contradicting the rules of logic cannot be called rigorous and epistemically equivalent (Duppe, 2010). Examples of erroneous interpretations are the so called theories of disequilibrium whose postulates violate axioms or assumptions that are equilibrium determinations in themselves: Disequilibrium points are only equilibrium points under new constraints.

In terms of our discussion, this means that just as no economic model is logically consistent if it is not in equilibrium, no dynamic model makes sense if it uses non-smooth rules of change or if its set X includes disequilibrium values. The economist's function must be, therefore, particular and limited to find an appropriate pair of mathematical objects (X and rules of change) that fit the formal structure. This corollary seems to be inherited nowadays by the Slutsky-Frisch-Tinbergen approach, a dominant methodological formalization in economic growth and macroeconomic models. Under this approach, shocks affecting any economy tend to be propagated in a muted fashion along the planning horizon because of the existence of filtration mechanisms that prevent the economy from continually wavering. Models that take the basis of this approach, such as business cycle models with linear specifications, set rules of change that restore disequilibrium in much the same way as mechanical oscillations models with damped motions do. That is to say; they are models that, after specifying the differential equation with perturbations, try to find the equilibrium trajectory with determined amplitude in some phases of the business cycle; as it happens with a differential equation that models the motion of a body with mass m suspended from a spring and subject to friction, resonance or external forces. This position is sharply criticized by other scholars who assert that even under the same assumptions of stable equilibrium and in the absence of exogenous shocks, the economy may oscillate indefinitely. In essence, trajectories of business cycles do not necessarily converge to stable attractors (Boldrin and Woodford, 1992).

As it is to be supposed, the adherence to a particular approach does condition the use of specific methods of dynamic optimization. Those who adhere to the Slutsky-Frisch-Tinbergen methodology assume that exogenous shocks do not destroy the toroidal resonant region and, therefore, that Hamiltonian equations remain integrable[7]. For them, shocks only affect the scale of the control variables, not their symplectic or conservative structure. Consequently, these authors have no difficulty in adopting some variants of the Turnpike Theorem to continue using the optimization and stability schemes of Hamiltonian mechanics, without altering the integrable nature of the equations. On the contrary, those who adopt the alternative approach think that shocks create zones in the phase spaces that are occupied by perturbed resonant toroids or fat fractals. They reject the tenets of the Turnpike Theorem by using Hamiltonians of disturbed systems or physical models of complex dynamics (Ramírez and Juárez, 2009)[8].

While the first approach has dominated the discipline for a long time, the second is still emerging. The relative importance of both approaches has been, however, changing as the dominant trends in ME have shifted

7 When a system loses its symplectic structure, two things can happen: (*i*) the system may regain its original structure through a transformation; or (*ii*) the system ceases to be integrable. In the first case, the transformation, called canonical, is achieved through a change of variable that preserves the original forms of the equations. In the second case, the constants of the integrable system vanish (Ott, 2002).

8 In some variants of the Turnpike Theorem, trajectories can produce cusps, Hopf bifurcations or chaos if the consumption preference parameter is high.

from the traditional Hilbert's formalist program to a paradigm based on the extensive use of simulated models. In the formalist program, economists have regarded it as essential to prove the existence of unique, stable equilibria in general equilibrium models deduced from unquestioned axioms (Weintraub, 2002). In contrast, simulation model-oriented economists have questioned the existence of these equilibria by adopting DST that make intensive use of digital computer programs, such as chaos or fractal theory (Weintraub, 2002).

3. ON THE REALISM OF ASSUMPTIONS

In addition to the previous reasons, there are others of particular nature explaining the differential expansion of DST in Economics. Two of them refer to the realism of assumptions and the conceptual validity of the translation of Economics into dynamic formats.

Lack of realism in assumptions is an essential point of the criticism of neoclassical dynamic models, and of the theory in general. Critics insist that it is untenable to draw valid conclusions from unrealistic or simplifying assumptions since, under these conditions, far from being successful, the deductive approach becomes misleading. The tendency to model dynamic systems with a high level of abstraction in their assumptions, sets Economics away from the usual practice of other sciences, consisting only of formalizing long-established results that are grounded on empirically-tested assumptions (Sarukkai, 2012). Contrary to this practice, Economics follows a similar route to Mathematics, in which it only models what can be reduced to a formal project, virtually ignoring the factual reliability of assumptions. In neoclassical dynamic models, assumptions do not necessarily need to have real descriptive content for the simple reason that they are only predictive tools. Hence, as a result of this instrumentalist view, it is not possible to expect the same unreasonable effectiveness of Mathematics in Economics as in other sciences (Velupillai, 2005). In this context, Sarukkai (2012) says that the relationship between Mathematics and Economy is one of subordination, not cooperation.

In order to make this point clearer, let us draw our attention to one assumption that is now very important in dynamic modeling: Bounded rationality. It aroused in the realm of the theory of organization as H.

Simon's reaction to the traditional view of modeling decision-making employing rational optimization (Barros, 2010). Unlike the original concept of ideal rationality in which economic agents are fully-informed maximizers of utility or profits, Simon placed the assumption of bounded rationality at the core of a different decision-making process. He says that an agent learns about his decisions in a search process guided by aspiration levels or values of variable goals (Selten, 1999). This process, named satisficing, is not fixed, as aspiration levels change with different situations: They can be raised or lowered depending on the ease of finding satisfactory alternative decisions.

These ideas of aspiration-adaptation gradually spread to many areas of Economics where decision making is a significant concern, in particular to Game Theory, where the assumption is currently of great importance either in mathematical theorizations (Evolutionary Game Theory) or in non-mathematical ones (Evolutionary Economics). As in many other parts of Economics, the meaning of bounded rationality depends on the dynamic view of authors. In Evolutionary Economics, the assumption helps model agents' behavior in a world with constant technological, organizational, and structural changes. In this world, there is a persistent emergence of innovations redefining economic structures and a complex dynamic involving nonlinear interactions (Witt, 2008). Therefore, the use of bounded rationality is more in the Simon tradition because agents are allowed to acquire information to obtain superior goals in an indeterminate process of satisficing. It hints that bounded rational decision making economic behavior has a non-optimizing character, but rather a flavor of continually adaptive learning. For this reason, evolutionary economists use the assumption as a means of finding evolutionarily stable strategies, in which new information enables the agents to search for more realistic options in calculating rational choices. This calculation has not to do with agents' perfect knowledge of a set of lotteries (Hodgson and Huang, 2010).

In neoclassical models, where bounded rationality is required, the situation is quite different. Unlike evolutionary economists, neoclassical use the assumption devoid of social considerations because the concept of dynamics has a strict quantitative meaning. Dynamic variables that are not measurable in terms of probability distributions or do not meet suitable convexity properties for optimization are expendable. As part of the mainstream, evolutionary game theorists give the assumption only a quantitative role in modeling boundedly rational economic behavior as optimizing. They are not interested in accounting for all the implications mentioned above but instead in finding an optimization method that allows calculating the optimal payoff rates in principal-agent models or the Bayesian-Nash pooling and separating equilibria in games with asymmetric information (Klaes, 2004).

These differences in perception have kept the two approaches apart, especially after evolutionary economists denied the existence of automatic adjustment mechanisms by which consumers or producers can relentlessly be a Bayesian or statistically adjusted rational maximizers of utility or profits. They say that people do not always obey Bayes rule as their probabilities judgments fail to meet the monotonicity requirements for the set of inclusion. In other words, they do not know how to choose the rational option when the situation is not familiar, and time is scarce. As a consequence of this, evolutionary economists have distrusted traditional methods of optimization based on Game Theory and opted for a more empirical approach using agent-based modeling (Selten, 1999).

The neoclassical economists' responses to these criticisms have been minimal as they consider that there are no common grounds for discussion. They insist that no argument about bounded rationality or any other assumption is valid if it does not fit a formal program. The reluctance to accept any kind of criticism is what critics consider a narrow idea of formalization or an inadequate translation of the language of Economics into Mathematics. Modeling what can only be expressed in equations means admitting that mathematics imposes its narrowness on economic analysis and, therefore, that 'the big picture —society's long-term transformation— is excluded of the analysis on the grounds that its dynamics cannot be sufficiently mathematized' (Hudson, 2000, p. 293).

4. THE TRANSLATION OF ECONOMIC CONCEPTS INTO DYNAMIC FORMATS

Regarding the conceptual validity of the translation of Economics into dynamic formats, we present two versions of the Malthusian population principle to illustrate how old ideas can be modified or updated using alternative techniques[9]. The idea behind this exercise is to show that each technique is subject to the concept involved (in this case, the principle of population) and, therefore, there is no established method to associate it with definitive mathematical tools or a neutral way of doing ME.

The first version is the widely known standard Malthusian model in which the stationary state is the inevitable destination of all possible trajectories of the population and means of subsistence. In its treatment, we use conventional DST. The second version is a new approach to the way oscillations can delay the convergence of those trajectories on the stable attractor. The study of oscillations is a hidden aspect of the principle, very little studied, that is at the core of Malthusian thinking, especially because oscillations, or retrograde and progressive movements experienced by the population's welfare around the "subsistence floor," are linked to Malthus's idea that a highly stratified society produces different demographic regimes. In its modeling, we use delay differential equations, which to our best knowledge, have not been used before to this purpose.

4.1. The traditional view of the principle

The typical Malthusian path of population growth assumes an economy that works under two assumptions. First, means of subsistence are determined by a production function $K(t)$ that depends on the population $P(t)$, the exogenous technological parameter $A > 0$, and the coefficient of decreasing marginal returns $0 < \alpha < 1$[10]. Second, the population grows logistically as an inverse function of the reciprocal of the *per capita* product $k(t) = K(t)/P(t)$, expanded by a constant s. This constant is the lower limit of the per capita product's growth rate. In formal terms:

---

9 Malthus defines the principle of population as the constant tendency of all animated life to increase beyond its stock of means of subsistence. In doing so he establishes three assumptions regarded as fixed laws of human nature: (*i*) food is necessary to the existence of man; (*ii*) passion between the sexes is inevitable and; (*iii*) the power of population is indefinitely greater than the power of the earth to produce subsistence for man. "From these assumptions, Malthus came immediately to his famous ratios and his thesis that strong and constant forces must necessarily hold the superior power of population over subsistence in check" (Dooley 1988, p. 200).

10 This is a standard function recently used by Pingle (2003).

$$\frac{dP(t)}{dt} = r_0 P(t)\left[1 - s\frac{P(t)}{K(t)}\right] \qquad [1]$$

where: $K(t) = AP(t)^{\alpha}$

To express equation [1] in terms of the *per capita* product, we assume that $k(t) = K(t)/P(t) = AP(t)^{\alpha-1}$ grows according to the differential equation:

$$k'(t) = A(\alpha-1)P(t)^{\alpha-2}\,P'(t) = (\alpha-1)k(t)\frac{P'(t)}{P(t)} \qquad [2]$$

After separating the variables and replacing $k(t)$ and [1] in [2], we have:

$$\frac{k'(t)}{(\alpha-1)k(t)} = r_0\left(1 - \frac{s}{k(t)}\right) \qquad [3]$$

$$k'(t) - r_0(\alpha-1)k(t) = -r_0 s(\alpha-1) \qquad [4]$$

This result eventually produces the trajectory of the *per capita* product:

$$k(t) = \left[k_0 - s\right]e^{r_0(\alpha-1)t} + s \qquad [5]$$

or expressed in terms of birth a and mortality b rates, with $a = r_0$ and $b = r_0 s$:

$$k(t) = \left[k_0 - \frac{b}{a}\right]e^{a(\alpha-1)t} + \frac{b}{a} \qquad [6]$$

Thus, if we introduce [5] or [6] into our definition of $k(t) = AP(t)^{\alpha-1}$ and solve $P(t)$ then we will obtain the equation for the target population:

$$P(t) = \left(\frac{k(t)}{A}\right)^{1/(\alpha-1)} \qquad [7]$$

Equation [7] shows that the population's trajectory will converge to a defined value by birth and mortality rates and the technological constant, in other words:

$$\lim_{t\to\infty} P(t) = P_e(t) = \left(\frac{s}{A}\right)^{1/(\alpha-1)} \quad [8]$$

since $\lim_{t\to\infty} k(t) = s$, and $s = b/a$.

Likewise, if we replace [7] in the production function and apply limits, we find that the $K(t)$ and $P_e(t)$ attractors are regulated by the same constants:

$$\lim_{t\to\infty} K(t) = K_e(t) = A\left(\frac{s}{A}\right)^{\alpha/(\alpha-1)} \quad [9]$$

The convergence of attractors in [8] and [9] will be faster as the value of α decreases when $t\to\infty$. In the limit case, the instantaneous rates of all variables will be nil[11]:

$$\lim_{t\to\infty} \frac{P'(t)}{P(t)} = \lim_{t\to\infty} \frac{K'(t)}{K(t)} = \lim_{t\to\infty} \frac{k'(t)}{k(t)} = 0 \quad [10]$$

4.2. The principle with delays and the existence of oscillations

Most neoclassical economists believe that the stationary state of equation [10] is the only possible result of the Malthusian population principle. We claim that a complete analysis of the principle requires introducing parameters into equation [1] that capture the presence of oscillations. One way to do so is assuming that population growth is not instantly affected by the birth rate but rather that there is a period of delay during which $P(t-\tau)$ influences $P(t)$ through mean birth and death rates. Since subsistence levels also affect population growth after a delay has elapsed, the non-linear effect created by oscillations in the means of subsistence is transmitted to the $P(t)$ variable through a delay in the inhibiting term of the logistic equation (Kuang, 1993):

[11] In the presence of diminishing returns, production will grow at a slower rate than the population creating a permanent drop in the *per capita* product. The resulting loss of welfare will in turn reduce population and production growth to the point where $k(t)$ converges to the stationary state.

$$\frac{dP(t)}{dt} = r_0 P(t)\left[1 - s\frac{P(t-\tau)}{K(t)}\right] \qquad [11]$$

where: $K(t) = AP(t-\tau)^{\alpha}$

The resulting Hutchinson-like equation has a known stable limit value ($P_s(t)$) provided that:

$$\lim_{t\to\infty} P'(t)/P(t) = 0, \text{ and } \lim_{t\to\infty} P(t) = \lim_{t\to\infty} P(t-\tau) = P_s(t).$$

In particular, Kuang (1993) shows that if $r_0 t < \pi/2$ then [11] will converge on a stable limit value, which in our case is the attractor $P_s(t) = (s/A)^{1/(\alpha-1)}$.

By comparing the attractors of systems [1] and [11], it is possible to conclude that both coincide but only in its limit value, since if the $r_0 t < \pi/2$ condition does not hold, then oscillations produced by [11] will alter the overall asymptotic stability of trajectories. Changes in the values of r_0 and τ will produce quasi-periodic behaviors or fluctuations in the $P(t-\tau)$ term (Gopalsamy, 1992)[12]. To stress the impact of r_0 on oscillations, we will assume that this behaves according to [12]:

$$r_0(t) = k_1 + \frac{k_2}{1 + e^{a_1 + a_2 t}} \qquad [12]$$

where k_1 and k_2 are, respectively, the lower and upper asymptotes of the birth rate; a_1 is the birth level and a_2 the speed of change of $r(t)$. The new equation becomes:

$$\frac{1}{P(t)}\frac{dP(t)}{dt} = \left[k_1 + \frac{k_2}{1 + e^{a_1 + a_2 t}}\right]\left[1 - s\frac{P(t-\tau)}{K(t)}\right] \qquad [13]$$

The limit value of [13] is similar to that of [11] given that:

[12] The fluctuations produced by $P(t-\tau)$ are due to the effect of the changes in means of subsistence on the average birth and death rates in the interval $(t-\tau,t)$. Therefore, it is feasible to expect the population to oscillate around its point of equilibrium, depending on the variations in the means of subsistence.

$$\lim_{t\to\infty} r_0(t) = k_1;\ \lim_{t\to\infty} s(t) = s;\ \lim_{t\to\infty} P(t-\tau) = P(t);$$

$$\lim_{t\to\infty} K(t) = K = AP^{\alpha};\text{ and } \lim_{t\to\infty} A(t) = A.$$

The fundamental difference between [11] and [13] lies in the population's paths since the disturbance produced by $r_0(t)$ further accentuates the oscillations produced by the τ delay in the relation established by $k'(t)/k(t)$ and $P'(t)/P(t)$[13].

4.3. What does the introduction of the new DST reveal?

The exposition of the two previous models shows that there is no unique way to formalize the Malthusian population principle and, therefore, to find the equilibrium path. Different equilibrium paths need different DST to formalize them. In any case, the selection of a specific technique has advantages and disadvantages. A significant advantage of the smooth dynamic systems, like the first model, is that they provide unique stable equilibria in closed-formulas that make the numerical calculation of equations easier. This advantage, however, comes at a cost: They cannot discover, for example, the existence of oscillations. Similarly, an advantage of non-smooth dynamic systems, such as the second model, is to show that the steady-state equilibrium is only one result among many. The disadvantage is that the model cannot predict the equilibrium solution since oscillations are differential responses of populations to changes in their economic environment.

The rationale for choosing one or another DST is a matter that continues to be debated since there are always excesses and arbitrariness, not exempt from the ideological burden. As far as mainstream economists are concerned, they consider it pointless to use non-smooth dynamic systems because they assume that the stationary state is the only equilibrium possible to extract from Malthus's Essay. However, this a limited reading of that book, as his author repeatedly insisted on the need to consider the delays in the population's growth responses to means of

[13] The relation between those growth rates is observed in $k'(t)/k(t) = (A'(t)/A(t))+(\alpha-1)(P'(t)/P(t))$ that is obtained using [4] and $k(t)$.

subsistence growth. Far from looking for a stable equilibrium point, Malthus instead sought to emphasize the progressive or retrograde movements that differentiate the well-being of the population dedicated to different productive activities. Since not all the inhabitants experience these movements in the same way, there are no foundations to associate them with the same demographic behavior (Waterman, 1998). This demographic diversity cannot be shown with smooth dynamic systems that standardize the population's response to means of subsistence.

5. CONCLUSIONS

The paper argues that the use of DST has guided the growing mathematization of economics in recent decades. Not only has it made the dynamic analysis more complex on topics traditionally considered static, but it has also encouraged the development of new areas of knowledge and allowed substantiating little-studied results in Economics. However, there are flaws in how DST are applied. Specifically, the reductionism of the economic analysis to the formal program stands out. The adoption of ad hoc techniques by the neoclassical economists has led other authors to criticize the inclusion of assumptions and types of equilibria within the formats of mainstream Economics.

The overall conclusion of the paper is that any pairing between DST and Economics should be cautious because it is not realistic to assume that there is a general mathematical approach to Economics. Not all economic phenomena can be formalized or explained with equations. Nor is it true that there is always a unique way to model a phenomenon. For these reasons, it is important to decide in which sense the mathematization is useful to enrich the explanation of the economic problem at stake. Otherwise, economists will continue to perpetuate preconceived and abstract schemes in which form generally takes precedence over economic content. ◀

REFERENCES

Afonso, O. (2001). The impact of international trade on economic growth [Working Paper no. 106]. Faculdade de Economia, Universidade do Porto, Portugal.

Arrow, K., and Intriligator, M. (1991). Historical Introduction. In: K.J. Arrow and M. Intriligator (eds.), *Handbook of Mathematical Economics I*. New York: North-Holland.

Arthur, W., Durlauf, S., and Lane, D. (1997). Introduction. In: W.B. Arthur, S.N. Durlauf, and D.A. Lane (eds.), *The Economy as an Evolving Complex System II*. Redwood City: Addison-Wesley.

Backhouse, R. (1998). If mathematics is informal, then perhaps we should accept that economics must be informal too. *The Economic Journal*, 108, pp. 1848-1858. https://doi.org/10.1111/1468-0297.00380

Backus, D., Kehoe, P., and Kydland, F.E. (1992). International real business cycles. *Journal of Political Economy*, 100. pp 745-775. http://dx.doi.org/10.1086/261838

Bali, S. (2011). Econophysics, thermoeconomics, and phynance. *The Journal of International Social Research*, 4(18), pp. 379-388. https://doi.org/10.17719/jisr.20111811038

Barros, G. (2010). Herbert A. Simon and the concept of rationality: Boundaries and procedures. *Brazilian Journal of Political Economy*, 30-3(119), pp. 455-472. https://doi.org/10.1590/S0101-31572010000300006

Beed, C., and Kane, O. (1991). What is the critique of the mathematization in economics? *KYKLOS*, 44(4), pp 581-662. https://doi.org/10.1111/j.1467-6435.1991.tb01798.x

Bhatia, N., and Szego', G. (2002). *Stability Theory of Dynamical Systems*. Berlin: Springer-Verlag.

Binmore, K., Piccione, M., and Samuelson, P. (1998). Evolutionary stability in alternating-offers bargaining games. *Journal of Economic Theory*, 80, pp. 257-291. https://doi.org/10.1006/jeth.1997.2387

Boldrin, M., and Woodford, M. (1992). Equilibrium models displaying endogenous fluctuations and chaos: A survey. In: J. Benhabib (ed.), *Cycles and Chaos in Economic Equilibrium* (pp. 303-326). Princeton: Princeton University Press.

Boylan, T., and O'Gorman, P. (2007). Axiomatization and formalism in economics. *Journal of Economic Surveys*, 21(3), pp 426-446. https://doi.org/10.1111/j.1467-6419.2007.00509.x

Day, R. (1983). The emergence of chaos from classical economic growth. *Quarterly Journal of Economics*, 98, pp. 201-203. https://doi.org/10.2307/1885621

Debreu, G. (1984). Economic theory in the mathematical mode. *The American Economic Review*, 74, pp. 267-278.

Debreu, G. (1991). The mathematization of economic theory. *The American Economic Review*, 81, pp. 1-7.
Dooley, P. (1988). Malthus on long swings: The general case. *Canadian Journal of Economics*, XX1(1), pp. 200-205. https://doi.org/10.2307/135220
Duppe, T. (2010). Debreu's apologies for mathematical economics after 1983. *Erasmus Journal for Philosophy and Economics*, 3(1), pp. 1-32. https://doi.org/10.23941/ejpe.v3i1.37
Gopalsamy, K. (1992). *Stability and Oscillations in Delay Differential Equations of Population Dynamics*. Dordrecht: Kluwer Academic Publishers.
Hirsch, M., Smale, S., and Devaney, R. (2004). *Differential Equations, Dynamical Systems, and an Introduction to Chaos.* San Diego: Academic Press.
Hodgson, G., and Huang, K. (2012). Evolutionary game theory and evolutionary economics: Are they different species? *Journal of Evolutionary Economics*, 22, pp. 345-366. https://doi.org/10.1007/s00191-010-0203-3
Holt, R., Rosser, B., and Colander, D. (2011). The complexity era in economics. *Review of Political Economy*, 23(3), pp. 357-369. https://doi.org/10.1080/09538259.2011.583820
Howitt, P. (1999). Steady endogenous growth with population and R&D inputs growing. *Journal of Political Economy*, 107, pp. 715-730. https://doi.org/10.1086/250076
Hudson, M. (2000). The use and abuse of mathematical economics. *Journal of Economic Studies*, 27, pp. 292-315. https://doi.org/10.1108/01443580010341754
Kailatzidakis, P., Mamuneas, T., and Stengos, T. (2011). An updated ranking of academic journals in economics. *Canadian Journal of Economics*, 44(4), pp. 1525-1538. https://doi.org/10.1111/j.1540-5982.2011.01683.x
Keller, W. (2002). Geographic localization of international technology diffusion. *The American Economic Review*, 92, pp. 120-142. https://doi.org/10.1257/000282802760015630
Klaes, M. (2004). Evolutionary economics: In defense of "vagueness". *Journal of Economic Methodology*, 11(3), pp. 359-376. https://doi.org/10.1080/1350178042000252992
Kuang, Y. (1993). *Delay Differential Equations with Applications in Population Dynamics*. London: Academic Press.
Lorenz, H. (2009). Complexity in economic theory and real economic life. *European Review*, 17(2), pp. 403-421. https://doi.org/10.1017/S1062798709000799
Mirowski, P. (1989). *More Heat than Light. New York*: Cambridge University Press. https://doi.org/10.1017/S0266267100000559

Mitra, T., and Nishimura, K. (2001). Introduction to intertemporal equilibrium theory: Indeterminacy, bifurcations, and stability. *Journal of Economic Theory,* 96, pp. 1-12. https://doi.org/10.1006/jeth.2000.2763

Noldeke, G., and Van Damme, E. (1990). Signaling in a dynamic labor market. *Review of Economic Studies*, 57, pp. 1-23. https://doi.org/10.2307/2297540

Ott, E. (2012). *Chaos in Dynamical Systems*. Second Edition. Cambridge: Cambridge University Press. https://doi.org/10.1017/CBO9780511803260

Pingle, M. (2003). Introducing dynamic analysis using Malthus's principle of population. *Journal of Economic Education*, pp. 3-20. https://doi.org/10.1080/00220480309595196

Ramírez, J., and Juárez, D. (2009). La nueva escalada matemática. El impacto reciente de la teoría de sistemas dinámicos en economía. *Economía Mexicana*. Nueva Época, 18(1), pp. 71-103.

Sarukkai, S. (2012). Mathematics in economics: Reducibility and/or applicability? *New Mathematics and Natural Computation*, 8(1), pp. 81-93. https://doi.org/10.1142/S1793005712400042

Selten, R. (1999). What is bounded rationality? [SFB 303, Discussion Papers B-454]. Paper prepared for the Dahlem Conference 1999.

Varian, H. (1991). Dynamical systems with application to economics. In: K.J. Arrow and M. Intriligator (eds.), *Handbook of Mathematical Economics I*. New York: North-Holland.

Velupillai, K. (2005). The unreasonable ineffectiveness of mathematics in economics. *Cambridge Journal of Economics*, 29, pp. 849-872. https://doi.org/10.1093/cje/bei084

Velupillai, K. (2011). Non-linear dynamics, complexity, and randomness: Algorithmic foundations. *Journal of Economic Surveys*, 25(3), pp. 547-568. https://doi.org/10.1111/j.1467-6419.2010.00668.x

Waterman, A. (1998). Malthus, mathematics, and the mythology of coherence. *History of Political Economy*, 30(4), pp. 571-599. https://doi.org/10.1215/00182702-30-4-571

Weintraub, R. (1991). *Stabilizing Dynamics: Constructing Economic Knowledge*. Cambridge: Cambridge University Press.

Weintraub, R. (2002). *How Economics Became a Mathematical Science*. Durham: Duke University Press.

Witt, U. (2008). What is specific about evolutionary economics? *Journal of Evolutionary Economics,* 18, pp. 547-575. https://doi.org/10.1007/s00191-008-0107-7