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# KINEMATICS AND VELOCITY ELLIPSOID OF THE K GIANTS

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RESUMEN

Para estudiar la cinemática de las gigantes K (clase de luminosidad III) se han utilizado los movimientos propios de la nueva reducción de van Leeuwen de los datos del catálogo de Hipparcos. En el estudio final, se consideraron 11,372 estrellas, de las cuales 880 tienen velocidades radiales conocidas. Mediante una programación semidefinida se obtuvieron valores para los parámetros cinemáticos (constantes de Oort) y simultáneamente, para los coeficientes del elipsoide de velocidades. Se obliga a que se cumpla la condición de que tanto la solución para la vecindad solar calculada a partir de los parámetros cinemáticos como la obtenida a partir del elipsoide de velocidades sean iguales. La solución nos da valores de $21.83 \pm 0.26$ km s$^{-1}$ para el movimiento solar, y $A = 13.08 \pm 1.72$ km s$^{-1}$ kpc$^{-1}$ y $B = -10.21 \pm 1.47$ km s$^{-1}$ kpc$^{-1}$ para las constantes de Oort, lo cual implica una velocidad local de rotación de $197.94 \pm 44.73$ km s$^{-1}$ si suponemos una distancia al centro galáctico de $8.2 \pm 1.1$ kpc. Para las dispersiones de velocidades obtenemos $\sigma_x = 50.58 \pm 0.99$ km s$^{-1}$, $\sigma_y = 42.42 \pm 1.13$ km s$^{-1}$ y $\sigma_z = 32.92 \pm 0.56$ km s$^{-1}$, con una desviación del vértice de $-7.^\circ 53 \pm 3.^\circ 97$.

ABSTRACT

To study the kinematics of the K giant stars (luminosity class III) use is made of proper motions taken from van Leeuwen's new reduction of the *Hipparcos* catalog. 11,372 stars, of which 880 have radial velocities, were used in the final study. Semi-definite programming solves for the kinematical parameters such as the Oort constants and simultaneously for the coefficients of the velocity ellipsoid. The condition that both the solution for the solar velocity calculated from the kinematical parameters and from the velocity ellipsoid calculation be the same is enforced. The solution gives: solar velocity of $21.83 \pm 0.26$ km s$^{-1}$; Oort's constant's, in units of km s$^{-1}$ kpc$^{-1}$, $A = 13.08 \pm 1.72$ and $B = -10.21 \pm 1.47$, implying a rotational velocity of $197.94 \pm 44.73$ km s$^{-1}$ if we take the distance to the Galactic center as $8.2 \pm 1.1$ kpc; velocity dispersions, in units of km s$^{-1}$, of: $\sigma_x = 50.58 \pm 0.99$, $\sigma_y = 42.42 \pm 1.13$, $\sigma_z = 32.92 \pm 0.56$ with a vertex deviation of $-7.^\circ 53 \pm 3.^\circ 97$.

*Key Words:* Galaxy: kinematics and dynamics — methods: numerical

## 1. INTRODUCTION

This paper continues a series on the kinematics and velocity ellipsoids of the giant stars (luminosity class III). Previously studied are the O-B5 (Branham 2006), the M (Branham 2008), and the B6-9 and A giants (Branham 2009). The K giants help fill the lacuna between the A and the M giants. van Leeuwen (2007) has recently re-reduced the *Hipparcos* raw data to produce a catalog with lower mean error than the original, which adds impetus to this study as it did for the B6-9 and A stars. A reader may wonder, however, why one should proceed one spectrum and luminosity class at a time. Why not do a solution that incorporates all of the spectrum luminosity classes? The answer entails not an attempt to reduce each study to the "minimum publishable unit", but rather to the fact that the calculations for each spectrum luminosity group involve considerable labor. The basic mathematical tool used, semi-definite programming (SDP), is computationally intensive and to study, therefore, one spectrum luminosity group at a time seems reasonable.

The methodology remains similar to that for the previous studies, although with one difference. Because no evidence exists to suggest that K stars form part of the Gould belt, no plane will be fit to the data to see if it is inclined to the Galactic plane. There appears to be a clean break between the O-B stars and the A stars regarding participation in the Gould belt; O-B stars have definite Gould belt members whereas the A stars have none nor do the M stars. One solves for the kinematics and velocity ellipsoid of the K giants by use of SDP, which offers the advantage that the solar motion calculated from the velocity ellipsoid must be the same as that calculated from the kinematical parameters. Nor is it necessary to use the same adjustment criterion for the two set of calculations: the kinematical parameters may be reduced by use of a least squares criterion whereas the velocity ellipsoid may be calculated with the robust $L_1$ criterion (minimize the sum of the absolute values of the residuals), or with the same $L_1$criterion for both.

This study also examines in greater detail the calculation of incompleteness factors. In their classical work *Statistical Astronomy* Trumpler & Weaver (1962) refer to two incompleteness factors, $K_1$, which compensates for the deficiency of proper motions in a parallax catalog compared with a proper motion catalog, and $K_2$, which corrects for the absence of proper motions nearly in the line of sight and thus not detectable in either a proper motion or a parallax catalog.

## 2. THE OBSERVATIONAL DATA

The proper motions and parallaxes used in this study were taken from van Leeuwen's version of the *Hipparcos* Catalog (2007), henceforth called simply the *Hipparcos* Catalog, the radial velocities from the Wilson (Nagy 1991) and Strasbourg Data Centre (Barbier-Brossat & Figon 2000) catalogs. van Leeuwen's catalog omits a few stars contained in the original catalog (ESA 1997). For those few stars the relevant data were taken directly from the original catalog. The equinox of the *Hipparcos* Catalog is J2000 and the catalog epoch is J1991.25. Stars listed as spectral class K, luminosity class III were extracted from the catalog. This resulted in a total of 11,989 stars. Given the number of stars, larger than that of the spectrum luminosity groups used previously, one might feel that the group could be subdivided into subgroups. This is not desirable, however, because only 5.6% of the stars are K5 or later, none are K9, and 56.8% are K0 or K1. The K giants, therefore, are highly skewed towards the earlier stars, and any subdivisions would contain highly disparate numbers of stars, deleterious to the confidence that can be placed in the solutions for the subgroups.

The star's HD number determined if either of the two radial velocity catalogs contained an entry for that particular star. Not all of the data could be accepted. Negative parallaxes were excluded as were parallaxes smaller than 1 mas because the Ogorodnikov-Milne (OM) model was used for the equations of condition (Ogorodnikov 1965). This model, valid out to about 1 kpc, should be adequate because the minimum parallax used in this study, 1 mas, corresponds to a distance of 1 kpc. Parallaxes smaller than 1 mas have such large mean errors that their inclusion seems unwarranted because of the uncertainty in their distances. Known multiple stars, flagged in the catalog, contaminate the proper motion by confusing orbital motion with genuine proper motion and were also excluded. And some of the solutions for the astrometric data in the catalog, also flagged, are substandard and were likewise be excluded. This left 11,372 K giants.

## 3. THE SPACE DISTRIBUTION OF THE K GIANTS

Let $x, y, z$ be rectangular coordinates with origin at the Sun: $x$ points towards the Galactic centre, $y$ is perpendicular to $x$ in the direction if increasing $l$, and $z$ is positive for positive Galactic latitude. From $\varpi$, the star's parallax, $l$, its Galactic longitude, and $b$, its Galactic latitude, we calculate

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{\varpi} \begin{pmatrix} \cos l \cos b \\ \sin l \cos b \\ \sin b \end{pmatrix} . \tag{1}$$

Figure 1 shows the distribution of the stars in space, Figures 2, 3, and 4 show the distributions in the $x$-$y$, $x$-$z$, and $y$-$z$ planes. There is little concentration towards the Galactic plane. Define a moment matrix, referred

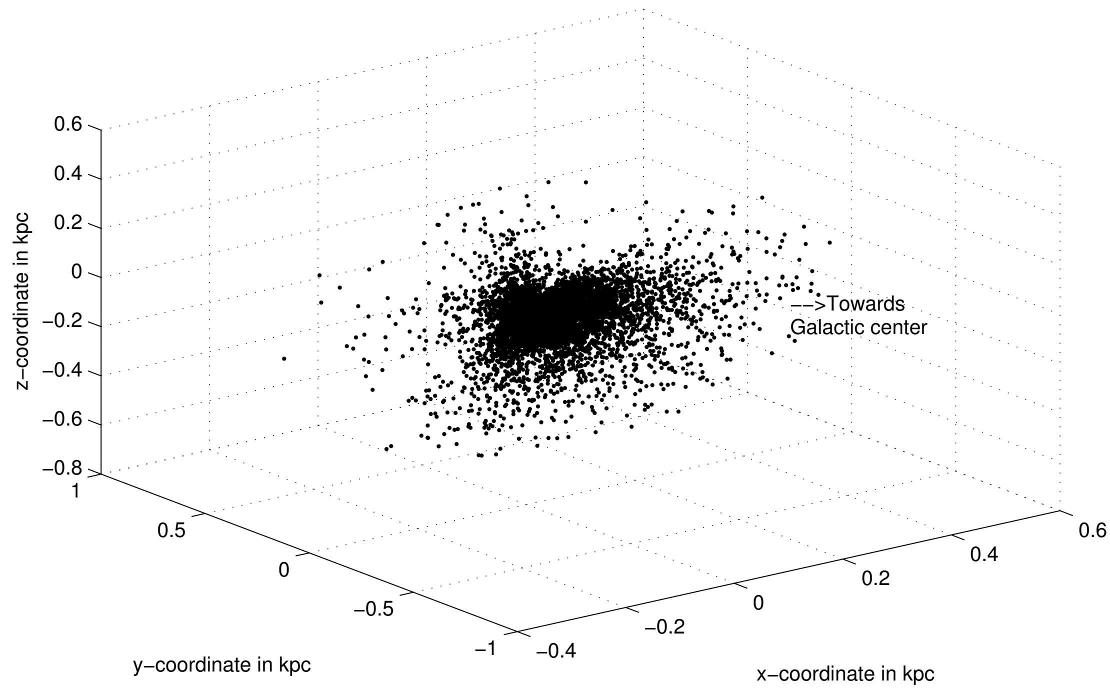


Fig. 1. Space distribution of K giants.

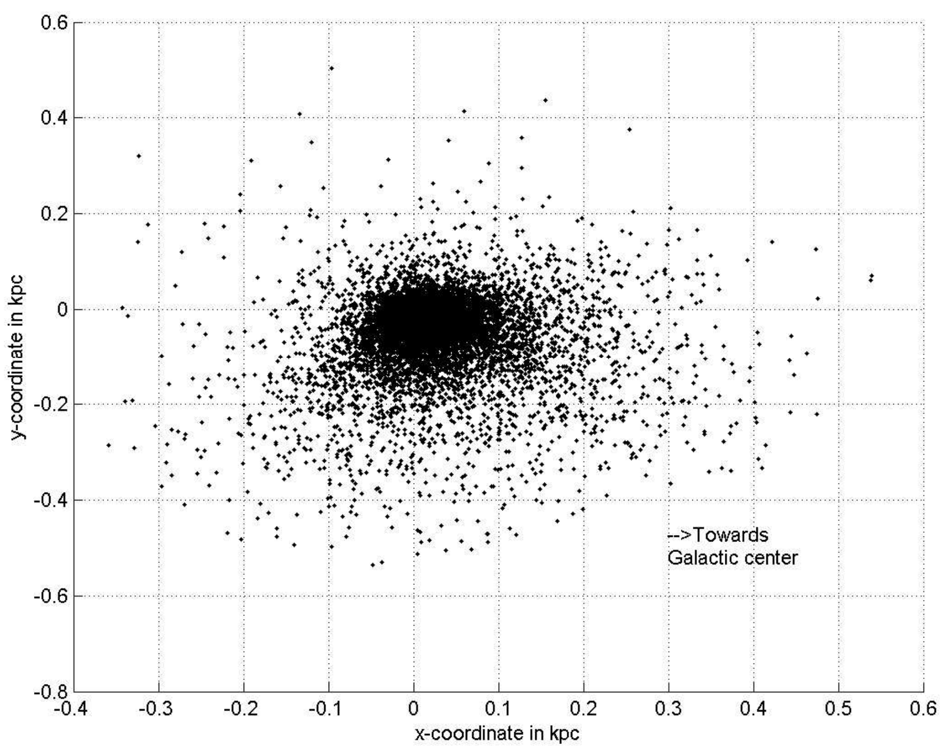


Fig. 2. Distribution in $x$-$y$ plane.

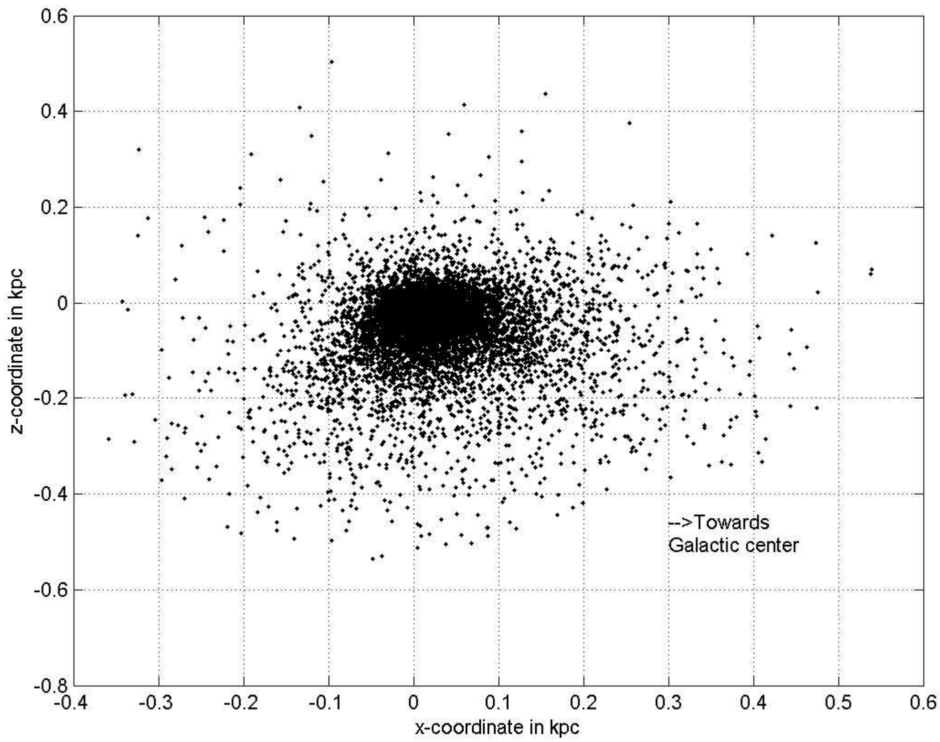


Fig. 3. Distribution in $x$-$z$ plane.

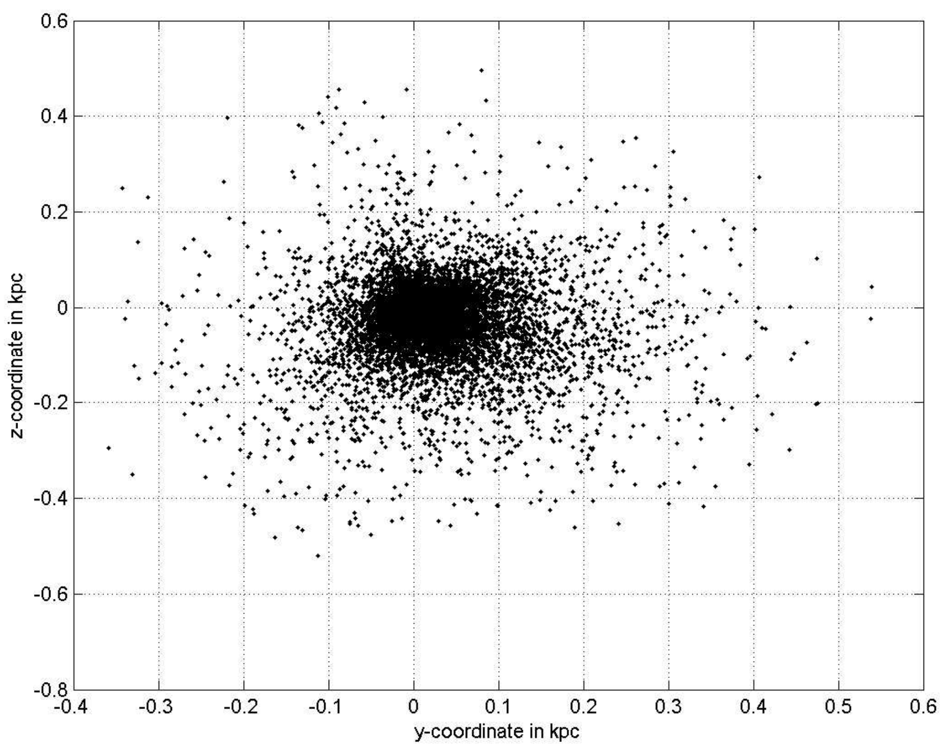


Fig. 4. Distribution in $y$-$z$ plane.

to the centroid of the distances, from the $x, y, z$ :

$$\begin{pmatrix} \sum_i (x_i-\bar{x})^2 & \sum_i (x_i-\bar{x})(y_i-\bar{y}) & \sum_i (x_i-\bar{x})(z_i-\bar{z}) \\ \sum_i (y_i-\bar{y})(x_i-\bar{x}) & \sum_i (y_i-\bar{y})^2 & \sum_i (y_i-\bar{y})(z_i-\bar{z}) \\ \sum_i (z_i-\bar{z})(x_i-\bar{x}) & \sum_i (z_i-\bar{z})(y_i-\bar{y}) & \sum_i (z_i-\bar{z})^2 \end{pmatrix} .$$

An eigenvalue-eigenvector decomposition of this matrix confirms this impression: the respective eigenvalues are 131.7, 115.1, and 95.1. The unit normal to the plane, defined as the normalized eigenvector associated with the z-direction, points to $b = 82.^\circ 107$. Although deviating somewhat from the North Galactic Pole, $b_G = 90^\circ$, the normal shows little evidence for strong Gould belt contamination because the pole of the Gould belt is located near 72°. Nor are the data strongly correlated: the correlation between $x$ and $y$ is $-2.0\%$, between $x$ and $z$ is $-3.4\%$, and between $y$ and $z$ is 6.2%. One may, therefore, consider the data homogeneous.

## 4. EQUATION OF CONDITION FOR KINEMATICS AND THE VELOCITY ELLIPSOID

Although the equations of condition have been given in my previous publications, they will be repeated here for the sake of completeness. Proper motion in Galactic longitude, $\mu_l$, and in Galactic latitude, $\mu_b$, follow from their counterparts in right ascension $\alpha$ and declination $\delta$, $\mu_\alpha$ and $\mu_\delta$, by use of the relations

$$\begin{aligned} \mu_l \cos b &= \mu_\alpha \cos\delta \cos\phi + \mu_\delta \sin\phi , \\ \mu_b &= -\mu_\alpha \cos\delta \sin\phi + \mu_\delta \cos\phi , \end{aligned} \qquad (2)$$

where $\phi$ is the Galactic parallactic angle.

Let the proper motion be measured in mas yr$^{-1}$, let $\dot{r}$ be the radial velocity in km s$^{-1}$, and $X, Y, Z$ the components of the reflex solar motion in km s$^{-1}$. Define the auxiliary parameters

$$\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} \cos l \cos b \\ \sin l \cos b \\ \sin b \end{pmatrix} ; \quad \begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix} = \begin{pmatrix} -\sin l \\ \cos l \\ 0 \end{pmatrix} ; \quad \begin{pmatrix} l_3 \\ m_3 \\ n_3 \end{pmatrix} = \begin{pmatrix} -\cos l \sin b \\ -\sin l \sin b \\ \cos b \end{pmatrix} .$$

The Ogorodnikov-Milne (OM) model was used for the equations of condition. See Ogorodnikov (1965) for a derivation of these equations. The OM model studies the motion of a group of stars whose centroid is located at distance $R_0$ from the Galactic center. $r$ is the distance from the centroid (the Sun) to the star, $V_0$ the reflex solar motion, and $V$ the velocity of the centroid at distance $R$ from the Galactic center. From elementary calculus we have

$$V = V_0 + D \cdot r , \qquad (3)$$

where $D$ is the displacement tensor of partial derivatives evaluated at $R_0$,

$$D = \begin{pmatrix} \partial V_x/\partial x & \partial V_x/\partial y & \partial V_x/\partial z \\ \partial V_y/\partial x & \partial V_y/\partial y & \partial V_y/\partial z \\ \partial V_z/\partial x & \partial V_z/\partial y & \partial V_z/\partial z \end{pmatrix}_{R=R_0} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} . \qquad (4)$$

Equation (4) involves a total of twelve unknowns, the three components of the reflex solar motion and the nine components of the displacement tensor. The equations of condition become (see Ogorodnikov 1965 for the details):

$$l_1^2 u_x + l_1 m_1 u_y + l_1 n_1 u_z + l_1 m_1 v_x + m_1^2 v_y + m_1 n_1 v_z + l_1 n_1 w_x + m_1 n_1 w_y + n_1^2 w_z - \varpi l_1 X - \varpi m_1 Y - \varpi n_1 Z = \varpi \dot{r} , \quad (5)$$

$$\sec\ b(-l_1 m_1 u_x - m_1^2 u_y - m_1 n_1 u_z + l_1^2 v_x + l_1 m_1 v_y + l_1 n_1 v_z) + \varpi l_2 X + \varpi m_2 Y + \varpi n_2 Z = \kappa \mu_l , \qquad (6)$$

$$\begin{aligned} &-\sec\ b[l_1^2 n_1 u_x + l_1 m_1 n_1 u_y + l_1 n_1^2 u_z + l_1 m_1 n_1 v_x + m_1^2 n_1 v_y + m_1 n_1^2 v_z + l_1(n_1^2-1) w_x + m_1(n_1^2-1) w_y \\ &-n_1(l_1^2+m_1^2) w_z + \varpi l_3 X + \varpi m_3 Y + \varpi - n_3 Z = \kappa \mu_b , \end{aligned} \qquad (7)$$

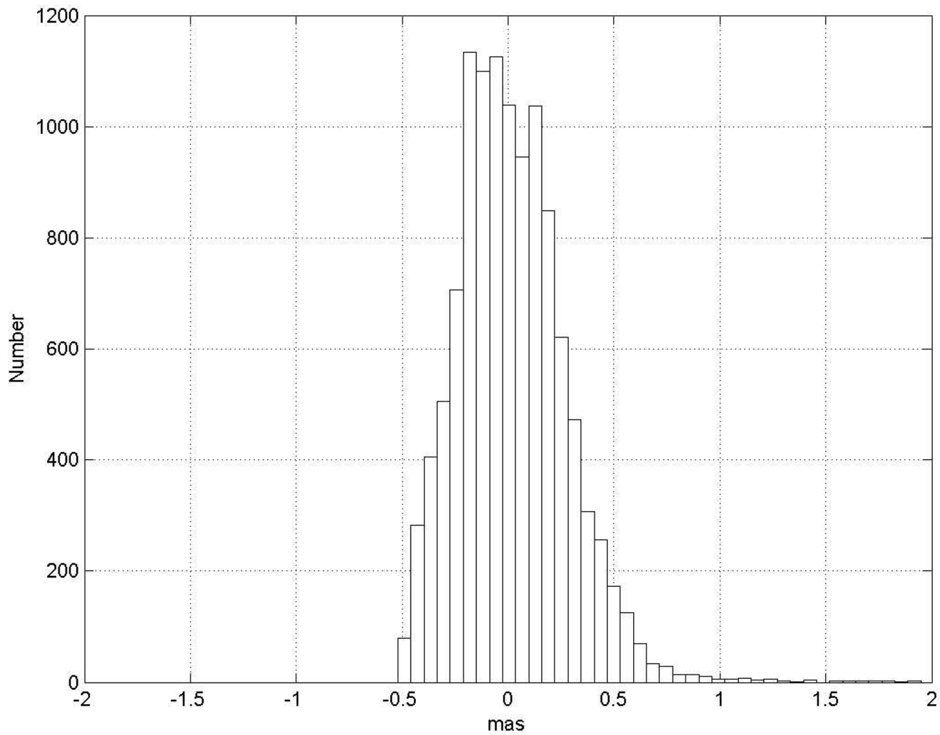


Fig. 5. Error distribution of the parallaxes.

where $\kappa$ is a conversion constant with value 4.74047 km s$^{-1}$ yr.

The equations as derived by Ogorodnikov (1965) actually use the distance $1/\varpi$ rather than the parallax $\varpi$ itself, but it is important to recast the equations to remove the parallax error from the denominator and thus ameliorate any possible Lutz-Kelker bias. Smith & Eichhorn (1996) have derived a procedure to correct the observed parallaxes, and this procedure was used to transform all of the parallaxes used in this study. Figure 5 shows the error distribution of the transformed parallaxes referred to the median of the parallaxes. Compared to a normal distribution this error distribution is somewhat skewed, coefficient of skewness of 0.48 versus 0 for the normal, more platykuritc, kurtosis of 1.63 versus 3 for the normal, and lighter tailed, Q factor of 0.27 versus 2.56 for the normal. These deviations, however, occur in the numerator and are thus less deleterious that would be deviations in the distance. The results presented in § 6 show that a runs test applied to the final residuals from equations (5–7) indicates complete randomness in the residuals and hence a satisfactory solution.

To calculate the velocity ellipsoid let $\dot{x}, \dot{y}, \dot{z}$ be the space velocities of a star. These are found from the proper motion and radial velocity:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sin l & -\cos l \sin b & \cos l \cos b \\ \cos l & -\sin l \sin b & \sin l \cos b \\ 0 & \cos b & \sin b \end{pmatrix} \cdot \begin{pmatrix} \kappa \mu_l \cos b/\varpi \\ \kappa \mu_b/\varpi \\ \dot{r} \end{pmatrix} . \tag{8}$$

The quadric surface to fit to these velocities becomes

$$a\dot{x}^2 + b\dot{y}^2 + c\dot{z}^2 + d\dot{x}\dot{y} + e\dot{x}\dot{z} + f\dot{y}\dot{z} + g\dot{x} + h\dot{y} + k\dot{z} + l = 0\,, \tag{9}$$

which can be rewritten as

$$\begin{pmatrix} \dot{x} - X & \dot{y} - Y & \dot{z} - Y \end{pmatrix} \cdot \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \cdot \begin{pmatrix} \dot{x} - X \\ \dot{y} - Y \\ \dot{z} - Z \end{pmatrix} = q\,. \tag{10}$$

To assure that the equation indeed corresponds to an ellipsoid one must impose the condition that the matrix be positive-definite and symmetric, $A = A^T$: $A_{11} = a$, $A_{12} = A_{21} = d/2$, $A_{13} = A_{31} = e/2$, $A_{22} = b$, $A_{23} = A_{32} = f/2$, and $A_{33} = c$. To avoid the trivial solution $a = b = \cdots = q = 0$ another condition must be imposed. The one I use is that the volume of the ellipsoid must be a maximum. Because the volume is proportional to the determinant of $A$, the condition becomes $\det(A) = \max$. An eigenvalue-eigenvector decomposition of the matrix $A$ yields the axes of the velocity ellipsoid and their orientation with respect to the Galactic coordinate system.

The solar velocity, $S_0 = \sqrt{X^2+Y^2+Z^2}$, calculated in equation (10) must be the same as the velocity found from equations (5–7). This condition can be imposed as part of an SDP formulation of the reduction problem. See Branham (2006) for details. Suffice to say that SDP minimizes the norm, whether least squares or $L_1$, of the residuals from equations (5–7), calculates the coefficients of the velocity ellipsoid, and imposes the conditions that the quadric surface of equation (9) must indeed be an ellipsoid and that the solar velocity must be the same from both the kinematical and the velocity ellipsoid calculations. For the calculations in this paper least squares reduced equations (5–7) and $L_1$ equation (10). This seems reasonable because the velocity ellipsoid in general shows more dispersion than the kinematical data.



## 5. SOME CORRECTIONS TO THE OBSERVATIONS AND COVARIANCE MATRICES

The total space motions needed in the velocity ellipsoid calculation should be corrected for the effects of Galactic rotation by modifying the proper motions and radial velocities used in the calculations to remove the rotation. This was done by the same procedure used in Branham (2006).

In theory one should also apply a correction for the incompleteness of the sample of the K giant stars taken from the *Hipparcos* catalogue. Trumpler & Weaver (1962) define a factor of incompleteness $K_1$ as

$$K_1 = N(m,\mu)/N_\varpi(m,\mu)\,, \tag{11}$$

where $N(m,\mu)$ is the number of stars in the sky for magnitude interval $m \pm \Delta m/2$ and proper motion interval $\mu \pm \Delta\mu/2$ and $N_\varpi(m,\mu)$ is the number of stars in the parallax catalogue for the same intervals. Equation (11) then transforms itself into a matrix with $\Delta m$ rows and $\Delta\mu$ columns. Trumpler & Weaver (1962) also define a second incompleteness factor $K_2$ to correct for stars of small proper motion. The evaluation of these factors becomes complicated, especially if one wishes to include mean errors, not discussed by Trumpler & Weaver (1962), and to base the evaluation of $K_2$ and its mean error on the ellipsoidal rather than the single star drift hypothesis.

Equation (11) is difficult to apply if there is insufficient overlap between the proper motion catalog and the parallax catalog. For the *Hipparcos* parallaxes a logical proper motion catalog would be the Tycho II catalog (Høg et al. 2000). Because the Tycho catalog gives no spectral types, one would have to use $(B-V)$ indices. The indices for the K giants fall into the range 1.06 to 1.65. Unfortunately, K dwarfs fall into the range 0.84 to 1.39 and the K supergiants into the range 1.42 to 1.94. Disentangling the luminosity groups could only be accomplished by assuming a certain ratio for supergiants to giants and giants to dwarfs and by also assuming a uniform distribution between the limits of the $(B-V)$ indices.

Before going to this trouble one should first query if it is really worth it. Two factors militate against such an effort: the sparse overlap between the parallax and the proper motion catalog with respect to magnitude, and the large mean errors for the calculated correction factor. Figure 6 shows contour plots of blue magnitude versus total proper motion for the Tycho and for the *Hipparcos* stars. The lack of overlap is manifest. In forming the table of equation (11), therefore, the correction factors become large because of the small number of *Hipparcos* stars in many of the cells.

That this is deleterious becomes evident is we calculate mean errors for the $K_1$ correction factors. Consider that the error in each $N(m,\mu)$ and $N_\varpi(m,\mu)$ cell is equal to the square root of the number of entries in that cell. Bok, following Öpik, calls this the "natural uncertainty" for the number (Bok 1937). Call these errors $\Delta N$ and $\Delta N_\varpi$. Then the error in the correction factor $K_1$ will be

$$\Delta K_1 = \left[ \begin{pmatrix} \Delta N & \Delta N_\varpi \end{pmatrix} \begin{pmatrix} 1/N_\varpi^2 & -N/N_\varpi^3 \\ -N/N_\varpi^3 & N^2/N_\varpi^4 \end{pmatrix} \begin{pmatrix} \Delta N \\ \Delta N_\varpi \end{pmatrix} \right]^{1/2} .$$

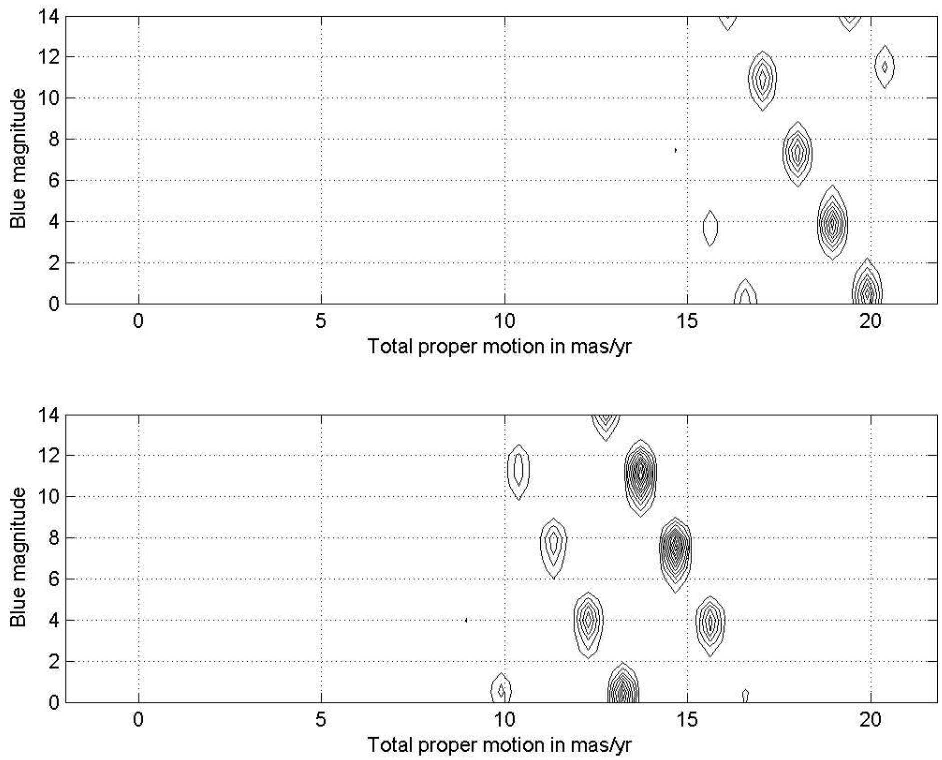


Fig. 6. Contour plots of blue magnitude versus total proper motion for Tycho and Hipparcos stars.

These errors are substantial. For the K stars, without discrimination as to luminosity class, the average $\Delta K_1$ is 10.3 for the cells containing information (many of the cells are empty) and the maximum 569. Do we really want to multiply the equations of condition by correction factors with such large mean errors? I feel that the homogeneity of the data is more important. Given that the K giants are homogeneous, we can say that they represent a random sample and $K_1$ corrections factors are superfluous.

One should, nevertheless, reconcile what has just been written with Trumpler and Weaver's statement that "this factor should not become very large". Seeing as they offer no detailed error analysis, the resolution of the conundrum would seem to be that they rely more on experience based on contemporary data sets, paltry in size compared to what is available today. In 1951, when their text was written, the Jenkinss parallax catalog had about 6,000 entries, and the largest proper motion catalog, Boss's *General Catalog,* about 33,000 entries. Today we deal with catalogs with ten to thirty times the number of entries, and one should accept with caution recommendations based on far fewer data.

Trumpler & Weaver (1962) also define a second incompleteness factor, $K_2$, to correct for the absence of proper motions in the parallax catalog nearly along the line of sight and hence undetectable. $K_2$ depends on the velocity ellipsoid. Trumpler and Weaver's treatment, while clear, also shows the limitations of numerical computing in 1951. For example, for the sake of simplicity they base their treatment on the single star drift hypothesis of stellar motion with respect to the local standard of rest rather than the more complete ellipsoidal hypothesis. I will base my treatment of the $K_2$ incompleteness factor on the ellipsoidal hypothesis and rather than divide the sky into finite cells centered on given values for $l$ and $b$, will integrate over Galactic longitude and latitude. And rather than divide the parallaxes into subgroups and calculating a $K_2$ for each subgroup, I will integrate over the parallaxes.

Let $t_l$ and $t_b$ be the tangential motions in $l$ and $b$, related to the proper motions by $t_l = \kappa\mu_l \cos b/\varpi$ and $t_b = \kappa\mu_b/\varpi$, and $t_{l0}$ and $t_{b0}$ the tangential motions induced by the solar velocity, $t_{l0} = -X \sin l + Y \cos l$, $t_{b0} = -X \cos l \sin b - Y \sin l \sin b + Z \sin b$. Then the bivariate distribution that represents tangential motion is expressed by

$$\Psi_{lb} = 1/2\pi\sigma_l\sigma_b\sqrt{1-\rho^2}\exp\left(-\left[(t_l-t_{l0})^2/\sigma_l^2 - 2\rho(t_l-t_{l0})(t_b-t_{b0})/\sigma_l\sigma_b + (t_b-t_{b0})^2/\sigma_b^2\right]/2(1-\rho^2)\right) . \tag{12}$$

In equation (12) $\sigma_l$ refers to the velocity dispersion in Galactic longitude, $\sigma_b$ the dispersion in latitude, and $\rho$ the correlation between the two. These quantities have to be calculated as ancillary parameters from the velocity ellipsoid calculations. I will present the relevant transformations given that the treatment in Trumpler & Weaver (1962) is only summary.

The eigenvalue-eigenvector decomposition of the matrix $A$ of equation (10) results in the diagonal matrix of the axes of the velocity ellipsoid,

$$\Lambda = \begin{pmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{pmatrix} ,$$

and the orthogonal matrix of the eigenvectors

$$V = \begin{pmatrix} V_1 & V_2 & V_3 \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} .$$

Let $s_1 = (X\,Y\,Z)$ be the vector of the components of the solar velocity, $s_2 = (-\sin\ l \quad \cos\ l \quad 0)$, and $s_3 = (-\cos\ l \sin\ b \quad -\sin\ l \sin\ b \quad \cos\ b)$. Define a vector $d_1 = s_2 - V_3$. Then $\gamma_1 = d_1/\sqrt{d_1 \cdot d_1}$. Likewise $d_2 = s_2 - V_2$ and $\gamma_2 = d_2/\sqrt{d_2 \cdot d_2}$. Finally,

$$\begin{aligned} \sigma_l^2 &= \sum_{i=1}^{3} \gamma_{1i}^2 \Lambda_{ii}^2 , \\ \sigma_b^2 &= \sum_{i=1}^{3} \gamma_{2i}^2 \Lambda_{ii}^2 , \\ \rho &= \sum_{i=1}^{3} \gamma_{1i}\gamma_{2i}\Lambda_{ii}^2/\sigma_l\sigma_b . \end{aligned}$$

The integral for the $K_2$ incompleteness factor can now be expressed as the quintuple integral

$$K_2 = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\int_{\varpi_{\min}}^{\varpi_{\max}}\int_0^{t_{l,c}}\int_0^{t_{b,c}} \Psi_{lb}\,dt_b\,dt_l\,d\varpi\,db\,dl \Big/ \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\int_{\varpi_{\min}}^{\varpi_{\max}}\int_0^{t_{l\,\max}}\int_0^{t_{b\,\max}} \Psi_{lb}\,dt_b\,dt_l\,d\varpi\,db\,dl . \tag{13}$$

In the upper integral $t_{l,c}$ and $t_{b,c}$ are cutoff values below which the tangential velocity cannot be detected. These values depend on $\mu_0$, the minimum detectable total proper motion: $\mu_0 = \sqrt{\mu_l \cos b^2 + \mu_b^2}$. Of the many ways to evaluate the integral, I opted for recursive 10th order Gaussian quadrature. The results of the evaluation will be given in the next section after the coefficients of the velocity ellipsoid have been calculated.

The covariance matrix is given in equation (25) of Branham (2006), and equation (26) of that publication shows how to calculate mean errors for quantities, such as the Oort constants, derived from the displacement tensor.

## 6. RESULTS

After the equations of condition for the kinematical parameters had been formed, I applied two checks for the adequacy of the reduction model. The first check simply calculates the singular values of the matrix of the equations of condition. An inadequate reduction model, for example one in which some unknowns are strongly correlated, results in a high condition number for the matrix because of small singular values. The condition number of the matrix of the equations of condition for the K giants, however, is low, 33.7. The second check calculates Eichhorn's efficiency (Eichhorn 1990), a parameter that varies from 0 to 1 with 0 indicating

TABLE 1

SOLUTION FOR KINEMATIC PARAMETERS FOR THE K III STARS

| Quantity | Value | Mean Error |
|---|---|---|
| $\sigma(1)$ (mean error of unit weight in mas km s$^{-1}$) | 103.64 | ··· |
| $u_x$ (in mas km s$^{-1}$) | 10.68 | 4.25 |
| $u_y$ (in mas km s$^{-1}$) | 21.84 | 2.15 |
| $u_z$ (in mas km s$^{-1}$) | −10.23 | 2.48 |
| $v_x$ (in mas km s$^{-1}$) | 1.41 | 2.41 |
| $v_y$ (in mas km s$^{-1}$) | −1.29 | 4.27 |
| $v_z$ (in mas km s$^{-1}$) | 8.47 | 2.57 |
| $w_x$ (in mas km s$^{-1}$) | −1.25 | 2.16 |
| $w_y$ (in mas km s$^{-1}$) | −4.39 | 1.98 |
| $w_z$ (in mas km s$^{-1}$) | 10.01 | 3.99 |
| $S_0$ (solar velocity in km s$^{-1}$) | 21.83 | 0.26 |
| $A$ (Oort constant in km s$^{-1}$ kpc$^{-1}$) | 13.08 | 1.72 |
| $B$ (Oort constant in km s$^{-1}$ kpc$^{-1}$) | −10.21 | 1.47 |
| $V0$ (Circular velocity in km s$^{-1}$) | 197.94 | 40.24 |
| $l_1$ (vertex deviation) | $-7.^\circ53$ | $3.^\circ97$ |
| $K$ (K term in km s$^{-1}$) | 4.69 | 3.87 |

TABLE 2

VELOCITY DISPERSION AND VERTEX DEVIATION OF THE K III STARS

| Quantity | Value | Mean Error |
|---|---|---|
| Mean absolute deviation of residuals in mas | 8.875 | ··· |
| $S_0$ (solar velocity in km s$^{-1}$) | 21.83 | 1.36 |
| $\sigma_1$ (velocity dispersion in x in km s$^{-1}$) | 50.58 | 0.99 |
| $\sigma_2$ (velocity dispersion in y in km s$^{-1}$) | 42.43 | 1.13 |
| $\sigma_3$ (velocity dispersion in z in km s$^{-1}$) | 32.92 | 0.56 |
| $l_1$ (longitude of $\sigma_1$) | $7.^\circ34$ | $7.^\circ14$ |
| $b_1$ (latitude of $\sigma_1$) | $6.^\circ14$ | $1.^\circ45$ |
| $l_2$ (longitude of $\sigma_2$) | $97.^\circ29$ | $5.^\circ61$ |
| $b_2$ (latitude of $\sigma_2$) | $-0.^\circ37$ | $2.^\circ06$ |
| $l_3$ (longitude of $\sigma_3$) | $190.^\circ79$ | $4.^\circ64$ |
| $b_3$ (latitude of $\sigma_3$) | $83.^\circ85$ | $1.^\circ20$ |

redundancy in the parameters and 1 that all parameters are necessary. The efficiency of 0.86 strongly indicates that *all* of the variables in the model are necessary and with little correlation among themselves.

The first solution for the K giants was calculated from all of the equations of condition in proper motion and 880 equations of condition in radial velocity. This solution calculated residuals needed to find discordant data. To eliminate the discordant data and perform a second solution I used a filter justified by previous experience: exclude a residual that exceeded five times the mean absolute deviation (MAD) of the residuals. This eliminated 432 equations of condition, a modest 1.83% trim.

Table 1 shows the solution for the kinematical unknowns, Table 2 for the coefficients and orientation of the velocity ellipsoid. Notice that both tables show the same solar velocity, although the mean errors are different,

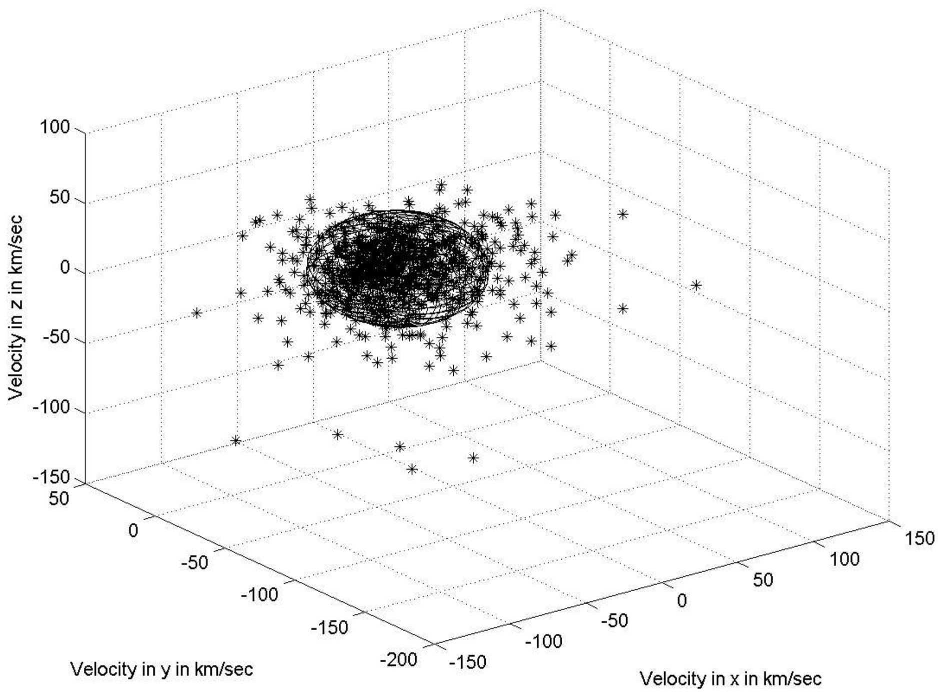


Fig. 7. Velocity ellipsoid for K giants; '*'=rectangular velocity of star.

as they should be because different residuals go into their calculation. For convenience the components for the displacement tensor are converted to the more familiar form of the solar motion, Oort constants, vertex deviation, and $K$ term. Also shown is the circular velocity $V_0$, found from the relation $V_0 = (A - B)R_0$, where $R_0$ is the distance to the centre of the Galaxy. Kerr & Lynden-Bell (1986) determine a value of $8.5 \pm 1.1$ kpc for $R_0$. Perryman (2008), however, after a survey of recent determinations feels that 8.2 kpc is a better determination, although he emphasizes its uncertainty: "...estimates for $R_0$ still lie in the broad range 7.5–8.5 kpc". The mean error for $V_0$ comes from the procedure given in Branham (2008) and uses 8.2 kpc for $R_0$ with the same mean error as given by Kerr and Lynden-Bell (1986).

The orientation of the velocity ellipsoid in space and in the $x$-$y$, $x$-$z$, and $y$-$z$ planes is shown in Figures 7–10.

## 7. DISCUSSION

The distribution of the residuals from the kinematical solution, after eliminating discordant residuals, as seen in the histogram of Figure 11 is somewhat skewed, coefficient of skewness 0.097, more platykurtic, kurtosis of 1.27, than the normal distribution, kurtosis of 3, and more lighter tailed, Hogg's Q factor of 0.35, than a normal distribution, Q=2.58. A runs test, however, reveals 11,606 runs out of an expected 11,596. The residuals, therefore, come from a random distribution.. This confirms, along with the singular values and Eichhorn's efficiency, that the reduction model suffers no serious defects and that inclusion of the $K_1$ incompleteness factor seems unnecessary.

For the residuals from the velocity ellipsoid the situation becomes different, as Figure 12 shows. The residuals are skewed, coefficient of skewness 2.35, highly leptokurtic, kurtosis 15.03, and light tailed, Q factor of 0.39. Nor are they even approximately random. The residuals for the K giants show 383 runs out of an expected 440. There is less than 0.02% probability that these residuals come from a normal distribution. This, however, hardly comes as a surprise. The ellipsoidal distribution of stellar velocities still remains only a crude approximation to the actual velocity distribution, even when the stars are divided into spectrum luminosity classes.

Having a solution for the coefficients of the velocity ellipsoid, one can discuss the significance of the $K_2$ incompleteness factor. The total proper motions for the K giants vary from a minimum of 0.149 mas to a

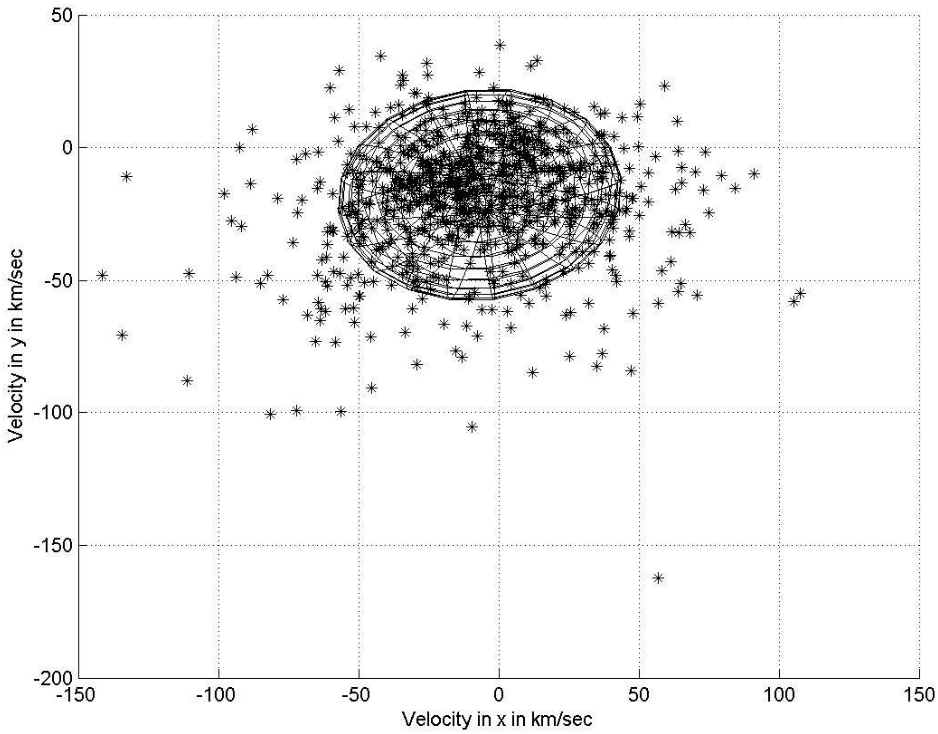


Fig. 8. Ellipsoid in $x$-$y$ plane.

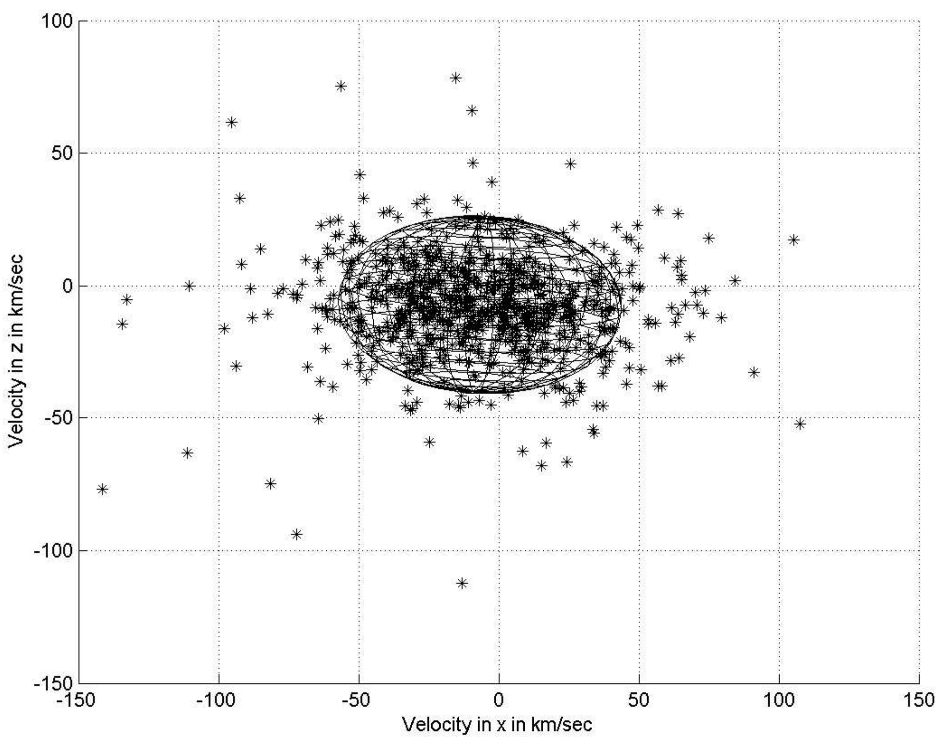


Fig. 9. Ellipsoid in $x$-$z$ plane.

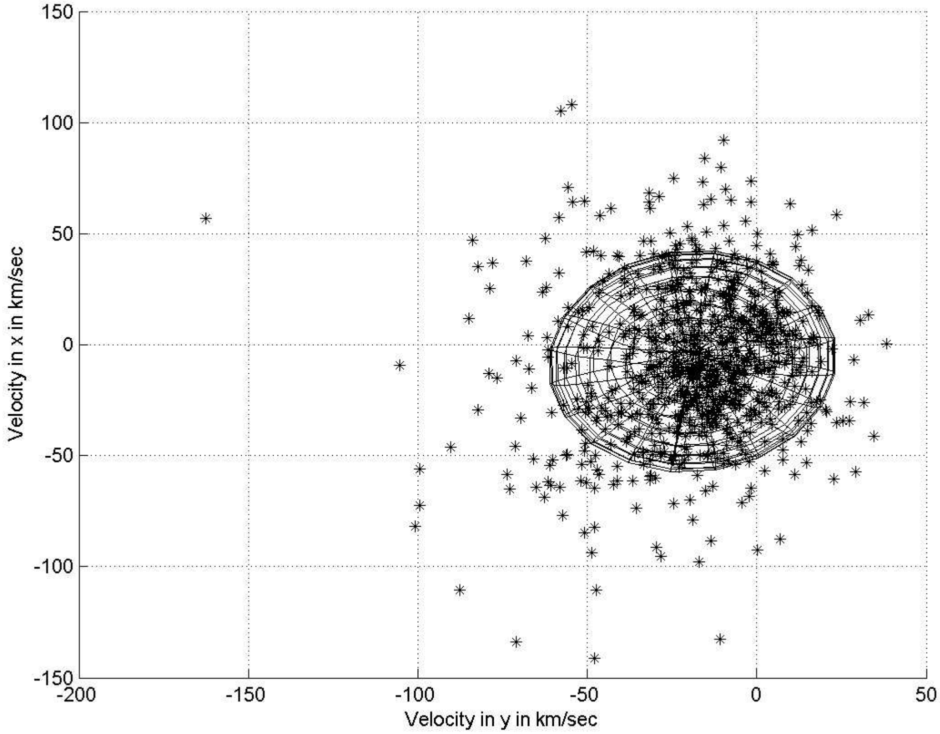


Fig. 10. Ellipsoid in $y$-$z$ plane.

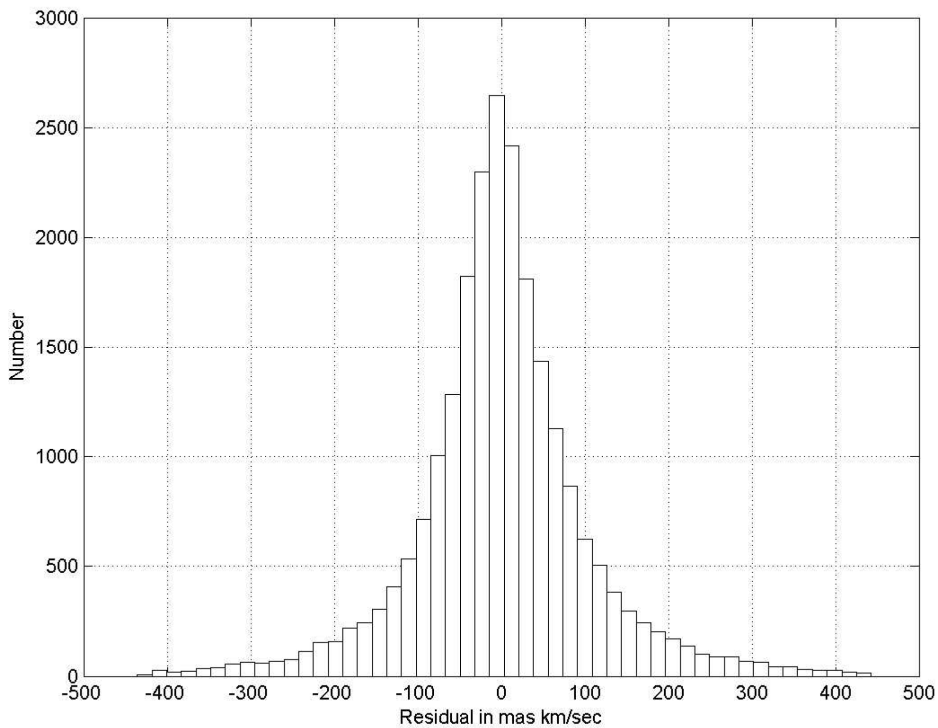


Fig. 11. Histogram of residuals from kinematical solution.

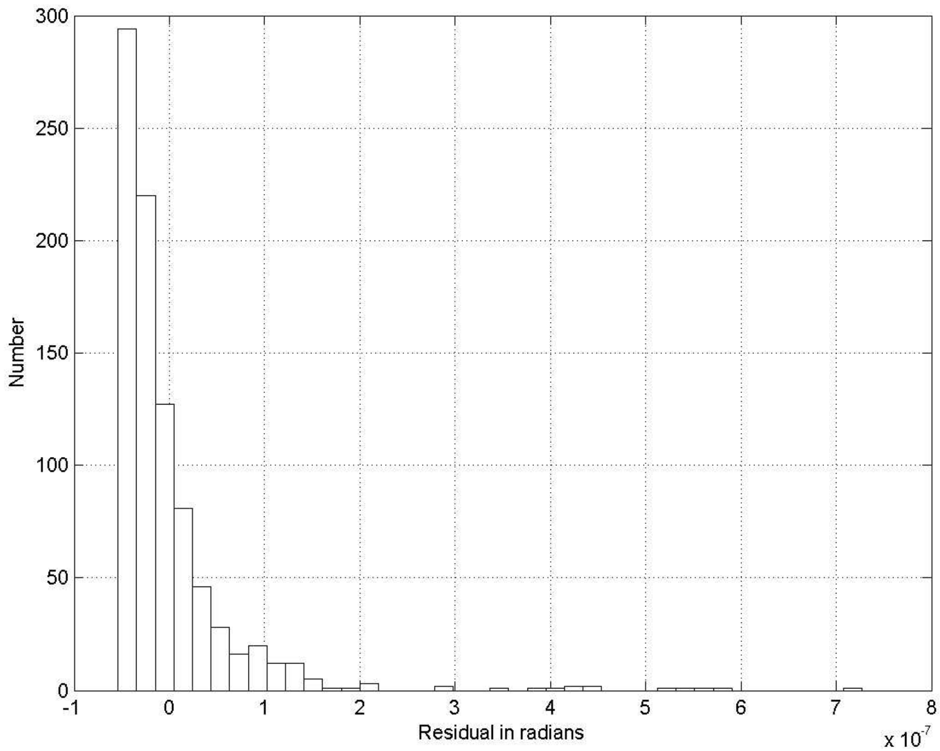


Fig. 12. Residuals in velocity for K giants.



maximum of 890 mas with a mean of 35.1 mas. Statistical outlier tests would consider most of the high total proper motions discordant although they are in fact most likely high velocity stars. These high total proper motions, nevertheless, must be eliminated from the lower integral in equation (13) to avoid artificially inflating the value of the integral. I chose a cutoff of 226 mas, based on Pierce's criterion (Branham 1990). The parallax limits were taken as 1 mas and 1000 mas. The evaluation of equation (13) showed that each integral, upper and lower, required 1,048,576 evaluations of Eq. (12) and that the value of equation (13) is less than $10^{-6}$. The number of evaluations shows why this type of quintuple numerical quadrature would have been impossible in 1951, when even a single numerical quadrature required considerable effort with a mechanical calculating machine. It also shows that the $K_2$ incompleteness factor becomes insignificant for the *Hipparcos* stars. The *Hipparcos* proper motion system is so sensitive that even proper motions nearly parallel to the line of sight can be detected.

In general one can say that the results given in Tables 1 and 2 follow the general tendencies shown by numerous previous studies: the solar velocity for the late stars is higher than that for the early stars; the dispersion of the velocity ellipsoid is higher for the later stars. For a survey of recent determinations of the kinematical parameters see Perryman (2008). The dispersions of the velocity ellipsoid are higher than those generally found, but this is a consequence of use of the SDP method; see Branham (2004). That the dispersions are higher for the later stars, however, is seen in Delhaye (1965), an old but still useful reference. The orientation of the velocity ellipsoid, Figures 6-9 shows no surprises.

The only quantity that shows a possibly discrepant value is the K term, putatively significant only for the early stars and with determinations falling near 5 km s$^{-1}$. The value in Table 1 seems high, but considering the size of the mean error one may question its relevance. My study of the B6-9 and A giants (Branham 2009), moreover, shows that the K term is sensitive to errors in the data and more difficult to determine than the $A$ and $B$ constants. One should, therefore, place little significance in the value calculated for the K term in Table 1.

## 8. CONCLUSIONS

Semi-definite programming proves itself once again a useful tool for problems of Galactic kinematics by allowing one to combine a solution for the kinematical parameters such as the Oort constants with one for the coefficients of the velocity ellipsoid. When applied to the K III stars the calculated solutions appear concordant with what others have found.

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