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A Note on the Bear-Hunter Problem in Mathematics
Una nota sobre el problema del oso y el cazador en Matemáticas

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Resumen

Revisamos un problema clásico en matemáticas sobre el oso y el cazador. Proponemos una aproximación diferente a los conjuntos numerables de soluciones conocidas para este problema y se presenta una nueva prueba de que éstas son todas las posibles soluciones. Finalmente, discutimos algunas consecuencias de la prueba, incluyendo las consecuencias del hecho de que no todas las trayectorias en el problema son geodésicas de la esfera.

Palabras clave: el cazador y el oso, geodésicas de la esfera.

Abstract

We revisit the classical problem in mathematics about the bear and the hunter; we take a different approach to the countable sets of known solutions giving a new proof that in fact these are all the solutions. We also discuss some consequences of the proof, in particular, of the fact that not all the paths involved in this problem are geodesics of the sphere.

Keywords: Bear-Hunter Problem, geodesics of the sphere.

THE PROBLEM

A classic problem in mathematics goes like this: a hunter sets up a camp from which to go bear hunting. Leaving camp, the hunter walked ten miles due south, then ten miles due west. At this point, he found a bear and shot it. He dragged the bear back to his camp, a distance of precisely ten miles. What color was the bear? (Costello, 1996).

Another formulation of the problem is: an explorer on the surface of the earth (assumed spherical) sees a bear 100 yards due south. The bear then travels 100 yards due east while the explorer remains stationary. The explorer now fires a shot due south, which travels straight and true, and strikes and slays the bear. What color was the bear? In his article, B. L. Schwartz (1960) provides the solutions to the problem using a Riemannian covering of the sphere.

Finally, the most common formulation of the problem is: a tourist leaves his camp and travels 10 km south, then 10 km west, where he meets a bear. He then travels 10 km north and finds that he is back at his camp. What color is the bear? (Maths in action Group, 1994) This formulation also appears in Chern (2001), where H. Whitney tell us a story about this problem:

> "I asked him [W. V. D. Hodge] one [question] that was recently going around Princeton: A man walked south five miles, then east five miles, then north five miles, and ended up where he had started. What could you say about where he had started? (Or more popularly, what color was the bear?) He insisted it

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must be the North Pole, and proceeded to give a careful proof; but I got the sense he did not really believe his proof was correct. (Try Antarctica.)"

The Classical Solution

The classical solution is that the bear is white (Gardner, 1994). The reason, in any formulation of the problem, is that the starting point is the North Pole and all is happening in a neighborhood of this point.

Although a Google Search gives a lot of sources, blogs and videos about this classic problem, no progress has been done on this problem since the analysis by B. L. Schwartz (1960).

Other Solutions

Other solutions that are known are the parallels 10 km above those with length $10/n$ km, where n is a natural number. It is easy to see that when the east movement completes exactly n turns around this parallel the conditions of the problem are satisfied (Schwartz, 1960).

So, we formulate the classical problem as follows.

Describe all points P in the 2-sphere with the round metric of radius R=6371 km such that moving P 10 km to south, then 10 km to the east and then 10 km to the north, we get P back to the starting point.

Of course, the color of the bear is not important and we focus on the mathematical arguments that explain which are the possible solutions of this problem.

Our analysis of the Solutions

We proved that the North Pole and the parallels we mentioned earlier are all the solutions. Define three functions on the sphere, using cylindrical coordinates $(t, e^{i\theta}) \in [0, \pi R] \times S^1$:

1) S, the function that takes a point P and yields a point 10 km south from P. That is, in coordinates, $(t, e^{i\theta}) \dashrightarrow (t+10, e^{i\theta})$.

2) N, the function that takes a point P and yields a point 10 km north from P. That is, in coordinates, $(t, e^{i\theta}) \dashrightarrow (t-10, e^{i\theta})$.

3) E, the function that takes a point P and yields a point 10 km east from P. That is, in coordinates, $(t, e^{i\theta}) \dashrightarrow (t, e^{i(\theta + 10/R\sin(t/R))})$.

We take the domain of the S function to be the sphere minus the North Pole and the South Polar cap, starting at the parallel 10 km north of South Pole. The North Pole is excluded, since otherwise the function would be multivalued. Also note that in the chosen South Polar cap, it is not possible to travel 10 km south. Similarly, for the domain of the N function, we take the sphere minus the South Pole and the North Polar cap, starting at the parallel 10 km south of North Pole. With this definitions, we have that both functions, S and N, are bijective functions onto their image. The domain of the E function is the sphere minus the North and the South Pole.

We already know that the North Pole is a solution (Gardner, 1994).

Now, the other solution points P in the sphere must be in the domain of S and satisfy the equation:

$$N(E(S(P)))=P \qquad (1)$$

Note that left side of the equation implies P is in the image of N. So, the points P in sphere we are interested in satisfy equation (1) and $P \in \text{Dom}(S) \cap \text{Im}(N)$. Since N is bijective onto its image, note that in $\text{Dom}(S) \cap \text{Im}(N)$ we have $N^{-1} = S$. Applying N^{-1} to the equation we obtain

$$E(S(P)) = N^{-1}(P) = S(P).$$

Then, we need to understand the fixed points of the E function in the range of S,

$$E(S(P)) = S(P).$$

But this E function is in fact a rotation of a circle, the parallel where $S(P)$ arrives. We know that rotations of S^1 have fixed points only when they make full n turns.

So, the solutions for the equation $E(S(P)) = S(P)$ are all points on the parallels with length $10/n$, where n is

a natural number. We have this type of parallels in the north hemisphere and in the south hemisphere. So the point *P* must be situated 10 km above this parallels. Existence of these points in the south hemisphere can be seen straightforwardly and we notice that we have countable number of parallels as solutions. For the north hemisphere we need to determine whether there is a point 10 km above a north parallel of length $10/n$.

Let $r=10/2\pi$ be the radius of one parallel of length 10 and let $R=6371$ be the radius of the sphere, the Earth. Then, the latitude is

$$\phi= \sin^{-1}(r/R)$$

and the distance from this parallel to the north pole is $R\phi \approx 1.59 < 10$. So, none of the parallel solutions can occur at the north hemisphere.

SOME GENERAL CONSEQUENCES OF THE PROOF

One can find in popular literature that the problem is related to spherical geometry or spherical trigonometry (Goldin, 1998; Klamkin, 1968; Van Brummelen, 2013). The proof presented earlier clarifies this misconception.

To make the north and south movements we use meridians, that is, geodesics of the sphere. The geodesics are the straight lines of the sphere in the sense that they minimize local distance. However, the movements to the east are being done through parallels, which are not geodesics of the sphere. The main characteristic of parallels is that they are orthogonal to the meridians. Hence, if we are allowed to use non geodesic orthogonal paths, we do not really need the geometry of the sphere to construct such paths that return to the original point. For example, as shown below, one can take polar coordinates in the Euclidean plane. That is, we are not using the geometry of the sphere, just a local behavior that can be reproduced on the Euclidean plane.

More precisely, consider the warped manifold I x S^1 (I an interval [a,b], with (b-a)>10, and S^1 parametrized by $e^{i\theta}$, θ in [0,2π)) with the metric $g=dt^2+\varphi^2(t)\,d\theta$, where φ is positive in the open interval (a,b), at least C^2, and satisfies initial conditions, $\varphi(a)=0$, $\varphi'(a)>0$.

We define, as before, the following functions:

1) *N*, displacement along *I*, towards *a*. That is (t, $e^{i\theta}$) ⋯⟫ (t-10, $e^{i\theta}$).
2) *S*, displacement along *I*, towards *b*. That is (t, $e^{i\theta}$) ⋯⟫ (t+10, $e^{i\theta}$).
3) *E*, displacement along S^1, for fixed t. That is (t, $e^{i\theta}$) ⋯⟫ (t, $e^{i(\theta + 10/\varphi(t))}$).

N is defined only on *[a+10, b)* x S^1. *S* is defined only on *(a,b-10)* x S^1. And E is defined on *(a,b)* x S^1.

If one considers *a=0, b=∞, φ(t)=t,* then we have the Euclidean plane. If $\varphi(t)= R\sin(t/R)$, we recover the example of the Sphere. We may also consider many other geometries, by choosing *φ(t)*; another typical example is *φ(t)=sinh(t)*, of course, the proof in the past section works also for these generalizations. And we have as solutions the point *(a, $e^{i\theta}$)* in *I x S^1*, and those points *(t-10, $e^{i\theta}$)* in *I x S^1* such that *n = 10/2πφ(t)*, is a natural number. Hence, one notes the following.

Corollary. If *φ* is non decreasing and *2πφ(10)>10*, then the solution is unique.

The Euclidean plane is included in this particular case.

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